ZERO DIVISOR GRAPHS OF CLASSES OF COMPLETELY PRIMARY FINITE RINGS OF MAXIMAL PRIME POWER CHARACTERISTIC

BY

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Declaration

I certify that this is my original work and that it has not been presented for a degree award in any other University.

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Dedication

To my late beloved father whose inspiration and love for education has forever stimulated my academic achievement and aspiration.

Abstract

It is well known that in a finite ring with identity, every element is either a zero divisor or a unit. The classification of finite rings is not fully settled. Different studies have generated interesting results on certain classes of finite rings. It is worthwhile to note that completely primary finite rings have proved to be useful towards the classification of finite rings. This is due to the fact that a finite ring has a unique maximal ideal if and only if it is a full matrix ring over a completely primary finite ring. Moreover, any commutative ring is a direct sum of completely primary finite rings. A deeper understanding of the elements in a finite ring enables us to fully understand the ring. In this study, we investigate and characterize the zero divisor graphs of classes of commutative completely primary finite rings of maximal prime power characteristic. For each class of rings, zero divisor graphs are drawn and trends in their geometric properties established through graph theoretic approach. In higher order cases, properties of zero divisors of commutative rings are employed in interpreting and determining the invariant geometrical structures of the graphs. This study has established that the diameter of the zero divisor graphs of the rings studied lie between 0 and 2 while their girth is either 3 or ∞. None of the rings has a zero divisor graph that is n-gon, where n is an integer greater than 3. Fundamentally, this study has revealed that rings whose zero divisor graphs are isomorphic are not necessarily isomorphic. The findings of this study extend further the knowledge about the structure theory of finite rings and in particular, the classification of the zero divisors of commutative completely primary finite rings.

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Notations

R .... a finite ring

$GR(p^{kr}, p^k)$ .... the Galois ring of order p^{kr} and characteristic p^k

R_0 .... the Galois ring of the form $GR(p^{kr}, p^k)$

J .... the Jacobson radical of the ring R

$R^\star$ .... the group of units of the ring R

$\mathbb{Z}$ .... the ring of integers

$\mathbb{Z}^+$ .... the set of positive integers

$\mathbb{Z}_n$ .... the ring of integers modulo n

p ....a prime integer p

$|A|$ .... the order of (the cardinal number of elements in) a set A

$\mathbb{F}_r$ .... a field of order r

$GF(p^r)$ .... Galois field of order p^r

$< a >$ ....the cyclic group generated by a

$o(a)$ .... the order of the element$a$ in the ring R

$charR$ .... the characteristic of the ring R

$Z(R)$ .... the set of zero-divisors of the ring R

$Z(R)^*$ .... the set of non-zero zero divisors of the ring R

$Z(R_0)^\star/\sim$ .... the set of equivalence classes of elements of $Z(R_0)^\star$

$R/Z(R)$ .... the quotient ring R modulo $Z(R)$

$d\left({}_RM\right)$ .... length of M as an $R-module$

(p) .... an ideal generated by p

$\Gamma(R)$ .... the zero- divisor graph of the ring R

$\Gamma_E(R)$ .... the zero-divisor graph of equivalence classes of the ring R

K_n .... a complete graph with n vertices

$K_{m,n}$ .... a complete bipartite graph with m and n vertices

$diam(\Gamma)$ .... the diameter of the graph Γ

$gr(\Gamma)$ .... the girth of a graph Γ

$\omega(\Gamma)$ .... the clique number of the graph Γ

$\chi(\Gamma)$ .... the chromatic number of the graph Γ

$b(\Gamma(R))$ .... the binding number of the zero divisor graph of R

S_{p^k} .... the Symmetric group of order $(p^k)!$

Chapter 1

Introduction

1.1 Background of the study

Rings considered in this thesis are finite, associative and commutative with 1 as the identity. Although finite rings have been studied extensively in recent years by Raghavendran [25] and Wilson [29] and tools necessary for describing completely primary finite rings have been available for some time (See [12, 13, 14, 15, 16] and [25]), their classification into well known structures is not complete. For any given ring R, the elements of R are either units or zero divisors. Classification of finite rings would be complete and better understood if the structures of both sets of elements in a ring are known. However, most research on rings has been on classification of their units. This perhaps is because a set of zero divisors, $Z(R)$ of any general ring R lacks algebraic structure. The set $Z(R)$ is not necessarily closed under addition. For instance in the ring $\mathbb{Z}_6$, 2 and 3 are zero divisors while $2+3$ is not. Thus $Z(R)$ is typically not a subring of R and hence is not an ideal. In 1988, Istvan Beck [11] introduced an alternative approach to the study of sets of zero divisors of rings using graph theory. His original definition of the graph of a commutative ring consist of the vertex set of all the elements of a ring such that distinct vertices x and y are adjacent if and only if $xy = 0$. Anderson and Livingston [7], later modified this definition. Their graph called the zero divisor graph of the ring R, denoted by $\Gamma(R)$, is a graph whose vertices are nonzero zero divisors of R such that two vertices u and v in $Z(R)\backslash\{0\} = Z(R)^\star$ are connected by an edge $u - v$, if and only if $uv = 0$. This definition, now considered standard, is the one adopted in our study. Properties of such a graph reveals important properties about the ring R. Classification of units of completely primary finite rings of prime power characteristic has been addressed by Chikunji and Oduor (See [12, 24]) among others. However, no evidence exists that similar studies have been conducted on the zero divisors of these rings. To bridge this gap, our study has addressed the structures of zero divisor graphs of three particular classes of completely primary finite rings of maximal prime power characteristic.

1.2 Statement of the problem and Justification

The structure of a finite ring with identity 1 is well understood if the characterization of its units and zero divisors is complete. Every finite ring with 1 is a direct sum of matrix rings over completely primary finite rings. A comprehensive understanding of completely primary finite rings would therefore make the classification of finite rings achievable. The units of the classes of completely primary finite rings considered in this thesis are well known (see [24]). However, the structures of the zero divisors of these rings have not been established. In this thesis, we study the geometric properties of graphs of the zero-divisors of completely primary finite rings of maximal prime power characteristic. Our study involves establishing the diameter, girth, the binding number, the clique number besides determining the partiteness of these graphs.

1.3 Objectives of the study

1.3.1 General objective

The structures of the zero divisors of certain classes of completely primary finite rings whose units have been established, [24] are not known. The objective of this study is to investigate and characterize the zero divisor graphs of commutative completely primary finite rings of maximal prime power characteristic using the graph theoretic properties.

1.3.2 Specific objectives

Specific objectives are to;

(i) identify and investigate the zero divisor graphs of the Galois rings.

(ii) characterize the zero divisor graphs of finite rings in which the product of a finite number of zero divisors is zero.

(iii) characterize the zero divisor graphs of finite rings in which the product of any two zero divisors lies in the Galois subring.

1.4 Basic Concepts On Ring Theory

Unless otherwise stated, the concepts and definitions in the sequel can also be found in [2, 20] and [21].

Definition 1.4.1 *A ring is a non empty set R together with two binary operations $+$ and $\times$ called addition and multiplication such that $(R, +)$ is an abelian group, R is associative under multiplication and the left and right distributive laws of multiplication over addition hold. R is a commutative ring if $ab = ba$ for all $a,\ b \in R$. It is called a ring with identity or with 1 if R contains the element $1 \neq 0$ such that $a.1 = 1.a = a$ for all $a \in R$.*

Definition 1.4.2 *A non empty subset H of a ring R is called a subring of R if H is itself a ring under the same operations of R. A subring I of a ring R is a left ideal if $rx \in I$ for all $r \in R$ and for all $x \in I$ and is called a right ideal if $xr \in I$ for all $r \in R$ and for all $x \in I$. If I is both a left and right ideal, then I is simply called an ideal of R.*

Definition 1.4.3 *Let S be a subset of a ring R and let $\{A_i : i \in I\}$ be the family of all ideals in R which contain S. Then $\bigcap_{i \in I} A_i$ is called the ideal generated by S and denoted by (S). The elements of S are called generators of the ideal (S). If $S = \{s_1, s_2, \ldots, s_n\}$, then the ideal (S) is denoted by $(s_1, s_2, \ldots, s_n)$ and said to be finitely generated. An ideal (s) generated by a single element is called a principal ideal.*

Definition 1.4.4 *An ideal M in a ring R is said to be maximal if $M \neq R$ and for every ideal N such that $M \subset N \subset R$, either $N = M$ or $N = R$.*

Definition 1.4.5 *For any ring R, an $R - Module$ is a set M together with two operations of addition in M and multiplication with elements of R such that for all $m,\ n \in M$ and $a,\ b \in R$ the following hold;*

(a) $(M, +)$ *is an Abelian group,*

(b) $(a + b).m = a.m + b.m$ *and* $a.(m + n) = a.m + a.n$,

(c) $(a.b).m = a.(b.m)$,

(d) $1.m = m$.

Definition 1.4.6 *Let R be a commutative ring with 1 and U be an R - module. The idealization of U over R is a ring $R \oplus U$ satisfying the following:* $(r_1, u_1) + (r_2, u_2) = (r_1 + r_2, u_1 + u_2)$ *and* $(r_1, u_1)(r_2, u_2) = (r_1 r_2, r_1 u_2 + r_2 u_1)$ *where* $r_1, r_2 \in R$ *and* $u_1, u_2 \in U$.

1.5 Units and zero divisors

A ring R is said to be *finite* if it contains a finite number of elements. Let R be a finite commutative ring with identity $1 \neq 0$. An element $u \in R$ is a *unit* if there exists an element $v \in R$ such that $uv = vu = 1 \neq 0$. An element $x \in R$ is a *zero divisor* if there exists a nonzero element $y \in R$ such that $xy = yx = 0$. A ring R is called a *division ring* or *skew field* if every nonzero element in R has a multiplicative inverse so that the nonzero elements form a group in R under multiplication. A *field* is a commutative ring with the identity element $1 \neq 0$ in which every nonzero element has a multiplicative inverse. A commutative ring with identity $1 \neq 0$ is called an *integral domain* if it has no zero divisors.

A *completely primary finite ring* is a ring R with identity $1 \neq 0$ whose subset of all its zero divisors forms the unique maximal ideal. A *Galois ring* is a finite ring with identity $1 \neq 0$ such that the set of all its zero divisors with 0 included forms a principal ideal. For instance, $\mathbb{Z}_{p^k}$, for some positive integer k, is a Galois ring with (p) as its unique maximal ideal. When $k = 1$, $\mathbb{Z}_{p^k} = \mathbb{F}_p$. The leading role in the classification of all the finite rings with identity certainly makes completely primary finite rings attractive to most researchers. Similar to completely primary finite rings so far studied, our attention has been restricted to the finite commutative rings in which the set of all the zero divisors forms an additive group.

Let R be an arbitrary ring (not necessarily rings considered in this thesis), then the set of all the zero divisors of R is not necessarily an ideal of the ring. For instance, the element $(2, 3)$ and $(1, 4)$ of the ring $\mathbb{Z}_4 \oplus \mathbb{Z}_6$ endowed with component-wise addition and multiplication are zero divisors, but if the set of the zero divisors were to be an ideal, then $(3, 1)$ would be a zero divisor, an obvious contradiction.

Theorem 1.5.1 *(See [15],Section 1) If a ring R is finite, then every left unit is a right unit and every left zero divisor is a right zero divisor. Furthermore, every element of R is either a unit or a zero divisor.*

Theorem 1.5.2 *[16] If a ring R has $n \geq 2$ left zero divisors (including zero), then R is a finite ring, and $|R| \leq n^2$.*

Proof.

Suppose $a \neq 0$ is a left zero divisor in R and consider the right ideal Ra of R. Since a is a left zero divisor in R, there exists $x \neq 0 \in R$ such that $ax = 0$, so that for all $r \in R$, $r(ax) = (ra)x = 0$. So Ra consists entirely of left zero divisors. Thus $|Ra| \leq n$. Now, since Ra is finite, consider the surjective additive group homomorphism $\phi : R \to Ra$ defined by $r \to ra$ with $ker\phi = \{y \in R : ya = 0\}$. We have $R/ker\phi \cong Ra$, and every element of the kernel is a left zero divisor of R (since $a \neq 0$), so that $|ker\phi| \leq n$. Thus $ker\phi$ and Ra are finite, so that R is finite and therefore, $|R| = |ker\phi||Ra| \leq n^2$. □

An ideal J of a ring R is said to be *nil* if all its elements are *nilpotent*. The ideal J is said to be *nilpotent* if $J^n = (0)$ for some $n \in \mathbb{Z}^+$. The ring of polynomials over a commutative ring R has been denoted by $R[x]$ while

$$J[x] = \{a_0 + a_1x + \cdots + a_sx^s, a_i \in J\} \subseteq R[x].$$

A polynomial $f(x) \in R[x]$ is called *monic* if the coefficient of the term with the highest power of x in $f(x)$ is equal to 1, which is the identity element of R. The group of automorphisms of R has been denoted by $AutR$ while Aut_TR denotes the subgroup of $AutR$ which fixes R elementwise. A ring R is called a *left* (or *right*) *Artinian* if any descending chain of *left* (or *right*) ideals of R has a minimal element and is an *Artinian* ring if it is both *left* and *right Artinian.* R is called *left* (or *right*) *Noetherian* if any ascending chain of *left* (or *right*) ideals of R has a maximal element and is said to be a *Noetherian* ring if R is both *left* and *right Noetherian.* The Jacobson radical $J(R)$ of a ring R is the intersection of all the maximal *left* (or *right*) ideals of R. It contains all the *left* and *right* nil ideals of R and if $u \in J(R)$, then $1 + u$ is a unit in R. By notation, $(J(R))^0 = R$ and $\bar{R} = R/J(R)$ so that for any $r \in R$, $\bar{r} = r + J(R)$. The characteristic of R is denoted by $charR$ and for any ring R, if $\bar{R} = R/J(R)$ is a division ring, then $char\ R = q$ where q is identically zero or $q = p^n$ for some prime integer p and positive integer n. Thus R has a copy of $\mathbb{Z}$ or $\mathbb{Z}_{p^n}$ where $\mathbb{Z}$ is the ring of integers while $\mathbb{Z}_{p^n}$ is the ring of integers modulo p^n respectively. An R-module M shall be both left and right such that $(as)b = a(sb)$ for each $a, b \in R$ and $s \in M$. For any ring R, all R-modules are unitary such that if M is an R-module, then $1m = m$ for each $m \in M$. A chain of submodules of an R-module M given by $M = M_0 \supset M_1 \supset \cdots \supset M_n = (0)$ is a *composition series* of R-submodules of M if the factor M_i/M_{i+1} has no proper submodules. A similar definition exists for the composition series of ideals.

Definition 1.5.3 *Let R be a ring. A positive integer $n \geq 1$ is called the index of nilpotency of $J(R)$ if $(J(R))^n = (0)$ and $(J(R))^{n-1} \neq (0)$.*

Theorem 1.5.4 *Let R be an Artinian ring. Then $J(R)$ is nilpotent.*

Proposition 1.5.5 *Let R be a ring and J be the Jacobson radical of R. Then $a \in R$ is a unit if and only if $a + J$ is a unit in R/J.*

Proof.

Clearly, if a is a unit in R, then $a + J$ is a unit in R/J. Conversely, let $a + J$ be a unit in R/J. Then there exists an element $b + J \in R/J$ such that $(a + J)(b + J) = (b + J)(a + J) = 1 + J$. Thus $ab = ba = 1$ which implies that $b = a^{-1}$. Hence a is a unit. □

Lemma 1.5.6 (Nakayama's Lemma) *Let I be an ideal of a ring R. The following conditions are equivalent*

(i) $I \subseteq J(R)$.

(ii) $1+i$ *is a unit for each* $i \in I$.

(iii) If M *is finitely generated* $R-$ *module such that* $IM = M$, *then* $M = (0)$.

(iv) If N *is a submodule of a finitely generated* $R-$ *module* M *such that* $M = IM + N$, *then* $M = N$.

Definition 1.5.7 *Let* R *be a ring and* M *an* $R-$ *module which has a composition series of* $R-$ *submodules* $M = M_0 \supset M_1 \supset \cdots \supset M_n = (0)$. *Then* n *is called the length of* M *as an* $R-$ *module and is denoted by* $d({}_RM)$.

Theorem 1.5.8 *(See [2]) Let* R *be a ring and* M *an* R*-module. If* $M = M_0 \oplus M_1 \oplus \cdots \oplus M_h$ *where* $M_i \neq (0)$ *for all* $0 \leq i \leq h$, $M = M'_0 \oplus M'_1 \oplus \cdots \oplus M'_s$ *where* $M'_j \neq (0)$ *for all* $0 \leq j \leq s$ *with* $h \leq s$ *and* $M_i \subseteq M'_j$, *for all* $0 \leq i \leq h$ *and for all* $0 \leq j \leq s$, *then* $M_i = M'_i$ *and* $h = s$.

Remark 1.5.9 *Let* R *be a ring,* M *an* $R-$ *module and* I *an ideal of* R *such that* $IM = (0)$. *Then* M *is an* $R/I\ -$ *module such that for each* $a + I \in R/I$ *and* $m \in M$, $(a+I)m = am$.

Lemma 1.5.10 *Let* R *be a ring with a finite length of composition series* $d({}_RR)$ *and* R_0 *a subring of* R *such that* $R = R_0 + J(R)$. *Then* $d({}_{R_0}R) = d({}_RR)$.

Definition 1.5.11 *Let* R *be a commutative ring,* σ *an automorphism of* R *and* P *the set of all left polynomials* $\Sigma_{i=0}^{s} a_i x^i$ *over* R. *Then* P *can be made into a ring by usual addition and the multiplication defined by the rule* $xa = \sigma(a)x$ *for all* $a \in R$. *This ring is called the skew polynomial ring over* R *given by* σ *and is denoted by* $R[x, \sigma]$. *If* σ *is of finite order* t, *then* $R[x,\sigma]$ *is denoted by* $R[x, \sigma, t]$.

Proposition 1.5.12 *Let $R[x,\sigma]$ be a skew polynomial ring over R and $g(x) \in R[x,\sigma]$ a monic polynomial. Then, for any $f(x) \in R[x,\sigma]$, there exists unique $q(x),\ r(x) \in R[x,\sigma]$ such that $f(x) = q(x)g(x) + r(x)$ with deg $r(x) <$ deg $g(x)$ or $r(x) = 0$.*

Remark 1.5.13 *Let $F[x,\sigma]$ be a skew polynomial ring over a field F. Then $F[x,\sigma]$ is a principal ideal domain.*

Proposition 1.5.14 *Let $R[x,\sigma]$ be a skew polynomial ring over R and $f(x) \in R[x,\sigma]$ a monic polynomial of degree r. Then $R[x,\sigma]\ / < f(x) > \cong \underbrace{R \oplus R \oplus \cdots \oplus R}_{h\ copies}$ as an R-module and hence $d(_R(R[x,\sigma]\ /\ <\ f(x)\ >)) = d(_RR).deg\ f(x) = rd(_RR)$.*

1.6 Local Rings

The concepts and definitions in this section are also adapted from [2] and [10].

1.6.1 Characterization of Local Rings

Let R be a finite commutative ring and let $R^\star$ denote the multiplicative group of units of R. Then R is local if it has a unique maximal ideal K and $1 + K \subseteq R^\star$. Thus, R is local if all the non units of R form an ideal.

Definition 1.6.1 *Let R be a ring, then R is called a local ring if its subset of non-units is closed under addition.*

Proposition 1.6.2 *(See [10], Theorem 2.3) For a ring R, the following statements are equivalent.*

(i) R is a local ring.

(ii) R has a unique maximal left ideal.

(iii) $J(R)$ is a maximal left ideal.

(iv) The set of elements of R with no left inverses is closed under addition.

(v) $J(R) = \{x \in R \mid Rx \neq R\}$.

(vi) $R/J(R)$ is a division ring.

(vii) $J(R) = \{x \in R \mid x \text{ is not a unit}\}$.

(viii) If $x \in R$, then either x or $1 + x$ is a unit.

Proposition 1.6.3 *Let R be an Artinian ring. If R has no non-trivial idempotent elements then, R is a local ring.*

Definition 1.6.4 *Let R be a ring and let S be the smallest subring of R containing the identity $1 \neq 0$ such that for any integer n, $n1$ is a unit in R, then if $(n1)^{-1} \in S$, then S is called the prime subring of R. If the prime subring S is a field, then S is called the prime subfield of R.*

Example 1.6.5 *Let p be a positive prime integer. Then $\mathbb{Z}_p[x]$ is ring of all polynomials with coefficients in $\mathbb{Z}_p$ as its prime subfield.*

Proposition 1.6.6 *(See [2]) Let R be a local ring with $J(R)$ a nil ideal and S the prime subring of R.*

(i) If $charR = 0$, then $S \cong \mathbb{Q}$.

(ii) If $charR = p^n$*, then* $S \cong \mathbb{Z}_{p^n}$*, for all* $n > 0$.

Proposition 1.6.7 *(See [2]) Let* R *be a local ring with* $J(R)$ *a nil ideal and* S *the prime subring of* R*. Then* $\bar{S} = (S + J(R))/J(R)$ *is a prime subfield of* $\bar{R} = R/J(R)$*.*

Definition 1.6.8 *Let* R *be a commutative local ring and* $f(x) \in R[x]$*. Then* $f(x)$ *is called a regular polynomial if* $f(x)$ *is not a zero divisor in* $R[x]$*.*

Theorem 1.6.9 *Let* R *be a commutative local ring with* $J(R)$ *a nil ideal and* $f(x) = \Sigma_{i=0}^{s} a_i x^i \in R[x]$*. Then* $f(x)$ *is nilpotent if and only if* $a_0, a_1, \ldots, a_s$ *are nilpotent.*

1.7 Galois Rings

It is well known (See Raghavendran [25],Theorem 1.7.1), that if R is a finite local ring, then $|R| = p^{nr}$, $|J(R)| = p^{(n-1)r}$, $\bar{R} = R/J(R) \cong GF(p^r)$ and $charR = p^k$, where $1 \leq k \leq n$, for some prime p and positive integers k, n, r.

Of special interest is the case when $k = n$. Then R is commutative and isomorphic to $\mathbb{Z}_{p^k}[x]/ < f(x) >$, where $f(x) \in \mathbb{Z}_{p^k}[x]$ is a monic irreducible polynomial of degree r in $\mathbb{Z}_p$. These rings, denoted by $GR(p^{kr}, p^k)$, are called *Galois rings* of order p^{kr} and characteristic p^k.

The following results due to Raghavendran and Wirt([25], [30]), are well known.

Proposition 1.7.1 *Let* R *be a Galois ring of the form* $GR(p^{kr}, p^k)$*. Then* $R = \mathbb{Z}_{p^k}[a]$ *where* a *is a root of monic polynomial* $f(x)$ *of degree* r *over* $\mathbb{Z}_{p^k}$ *which is irreducible over* $\mathbb{Z}_p$*.*

Proposition 1.7.2 *Let R be a finite local ring. Then R is Galois if and only if $J(R) = pR$ for some prime number p.*

Proposition 1.7.3 *[12] If R is a Galois ring, then $AutR \cong Aut\,(R/J(R))\,.$*

Lemma 1.7.4 *Let R be a Galois ring of the form $GR(p^{kr},\ p^k)$. Then R has a unique Galois subring of the form $GR(p^{kt},\ p^k)$ if and only if $t \mid r$.*

Remark 1.7.5 *A subring R_0 of a Galois ring R is not necessarily Galois. However, R_0 is Galois if and only if it is a principal ideal.*

Proposition 1.7.6 *Let R_0 and R_1 be two Galois rings of the same characteristic. Then $R_0 \cong R_1$ if and only if $R_0/J(R_0) \cong R_1/J(R_1)$.*

Proposition 1.7.7 *(see [12]) Let R be a Galois ring of order p^{kr} and of characteristic p^k, having a maximal ideal $J(R)$ such that $R/J(R) \cong GF(p^r)$.*

Let $\psi : R \to R/J(R)$ be the cannonical homomorphism and let $f \in \mathbb{Z}_{p^k}[x]$. If $f = a_0 + a_1x + \cdots + a_tx^t$, let $\psi(f)$ denote the polynomial $\psi(a_0) + \psi(a_1)x + \cdots + \psi(a_t)x^t$. Then if $\psi(f)$ is irreducible over $\mathbb{Z}_p$ and if $\bar{\lambda}$ is a root of $\psi(f)$ in $R/J(R)$, then there exists $\lambda \in R$ such that $\psi(\lambda) = \bar{\lambda}$ and $f(\lambda) = 0$. If in addition, R is commutative, then λ is uniquely determined by the given condition.

Proof.

Let $\lambda_0 \in \psi^{-1}(\bar{\lambda})$ and let R_0 be a commutative subring of R containing λ_0 (for instance the subring $\mathbb{Z}_{p^m}[\lambda_0]$). Then R_0 is a Galois ring with maximal ideal $R_0 \cap J(R) = (J(R_0))$. Let $\psi(f) = g \in \mathbb{Z}_p[x]$. Since g is irreducible over $\mathbb{Z}_p, \bar{\lambda}$ is a simple root of g and therefore $g(\bar{\lambda}) \neq \bar{0}$. Let $x \in \lambda_0 + (J(R_0))$. Then $\psi(f(x)) =$

$\psi(f)(\psi(x)) = \psi(f)(\bar{\lambda}) = 0$ and so $f(x) \in J(R)$, but $f(x) \in R_0$ since $x \in R_0$ and $\mathbb{Z}_{p^m}[x] \subset R_0$. Therefore $f(x) \in (J(R_0))$.

Now consider the map σ defined on $R_0 \oplus J(R_0)$ by $\sigma(\lambda_0 + u_i) = f(\lambda_0 + u_i)$ for all $\lambda \in R_0$ and $u_i \in J(R_0)$. We show that σ is injective. Let $\sigma(\lambda_0 + u_1) = \sigma(\lambda_0 + u_2)$ with $u_1, u_2 \in J(R_0)$. Then $f(\lambda_0 + u_1) - f(\lambda_0 + u_2) = 0$, and therefore, since R_0 is commutative, we use the binomial formula to obtain;

$0 = \Sigma_{i=1}^{t} a_i \left[(\lambda_0 + u_1)^i - (\lambda_0 + u_2)^i\right]$

$= \Sigma_{i=1}^{t} a_i \left[\Sigma_{j=0}^{i} \binom{i}{j} \lambda_0^{i-j} u_1^j - \Sigma_{j=0}^{i} \binom{i}{j} \lambda_0^{i-j} u_2^j\right]$

$= \Sigma_{i=1}^{t} a_i \left[\Sigma_{j=0}^{i} \binom{i}{j} \lambda_0^{i-j} (u_1^j - u_2^j)\right] = \Sigma_{i=1}^{t} a_i \left[\Sigma_{j=1}^{i} \binom{i}{j} \lambda_0^{i-j} (u_1^j - u_2^j)\right]$

$= (u_1 - u_2) \left[\Sigma_{i=1}^{t} a_i \left[\Sigma_{j=1}^{i} \binom{i}{j} \lambda_0^{i-j} \left(u_1^{j-1} + u_1^{j-2} u_2 + u_1^{j-3} u_2^2 + \cdots + u_2^{j-1}\right)\right]\right]$

$= (u_1 - u_2) \left[\Sigma_{i=1}^{t} i a_i \lambda_0^{i-1} + \Sigma_{i=2}^{t} a_i \left[\Sigma_{j=2}^{i} \binom{i}{j} \lambda_0^{i-j} \left(u_1^{j-1} + u_1^{j-2} u_2 + u_1^{j-3} u_2^2 + \cdots + u_2^{j-1}\right)\right]\right]$

$= (u_1 - u_2) \left(f'(\lambda_0) + u'\right),$

where $u' = \Sigma_{i=2}^{t} a_i \left[\Sigma_{j=2}^{i} \binom{i}{j} \lambda_0^{i-j} \left(u_1^{j-1} + u_1^{j-2} u_2 + u_1^{j-3} u_2^2 + \cdots + u_2^{j-1}\right)\right] \in (J(R_0))$.

Suppose $u_1 - u_2 \neq 0$, then $f'(\lambda_0) + u'$ is a zero divisor of R_0 so that

$\psi(f')(\bar{\lambda}) = \psi(f'(\lambda_0)) = 0$. But f' is the derivative of f and hence $\psi(f')$ is the derivative of $\psi(f)$, so $\psi(f')(\bar{\lambda}) = 0$ implies $\psi(f)$ has $\bar{\lambda}$ as a multiple root which is a contradiction since $\psi(f)$ is irreducible over $\mathbb{Z}_p$. Thus $f'(\lambda_0) \notin (J(R_0))$. So $f'(\lambda_0) + u'$ is invertible in R_0, implying $u_1 - u_2 = 0$, and σ is injective. But $|\lambda_0 + (J(R_0))| = |R_0|$, and $(J(R_0))$ is finite so that σ is surjective. Thus σ is onto, therefore there exist a unique $\lambda \in \lambda_0 + (J(R_0)) \subset \psi^{-1}(\bar{\lambda})$ such that $f(\lambda) = 0$. □

Corollary 1.7.8 *Let $f \in \mathbb{Z}_{p^k}[x]$, be a monic polynomial of degree r and $\psi(f)$ be irreducible over $\mathbb{Z}_p$. Then f has at least r roots in R. If R is commutative, then f has exactly r roots in R.*

The following result demonstrates the "existence" of Galois rings.

Proposition 1.7.9 *[25] Let $f \in \mathbb{Z}_{p^k}[x]$ be a monic polynomial of degree r whose image is irreducible over $\mathbb{Z}_p$. Then $R = \mathbb{Z}_{p^k}[x]/ < f(x) >$ is a Galois ring of order p^{kr} and characteristic p^k whose maximal ideal $J = pR$.*

Proof.

Consider the ideals $(p), (f) \subset (p, f) \subset \mathbb{Z}_{p^k}[x]$. Clearly,

$$R/pR = \left(\mathbb{Z}_{p^k}[x]/ < f(x) >\right) / \left((p, f)/ < f(x) >\right) \cong \mathbb{Z}_{p^k}[x]/ \left((p, f)\right) \cong GR(p^r, p).$$

So pR is a nilpotent ideal of R. Since R is commutative, then R/pR is invertible such that if $u \in pR$, then $u^n = 0$ for some $n \in \mathbb{Z}^+$. Let $w \notin pR$, then $w^m = 1+u$, $u \in pR$, for some $m \in \mathbb{Z}^+$, so $w^m = 1 - v, v \in pR$ and

$$w^m(1 + v + \cdots + v^{m-1}) = (1 - v)(1 + v + \cdots + v^{m-1}) = 1 - v^m = 1.$$

Therefore elements of R/pR are invertible and so R is a completely primary finite ring with maximal ideal pR. Since, f is monic of degree r, R has p^{kr} elements. Then R has order p^{kr}, R/pR has order p^r and characteristic of $R = p^k$. This implies that R is a Galois ring. □

Proposition 1.7.10 *[25] Given a prime integer p, and positive integers k and r, there exists a unique (up to isomorphism) Galois ring of order p^{kr} and $|R/pR| = p$.*

Proof.

Let R be a Galois ring of order p^{kr} and let $|R/pR| = p^r$. We establish that R is isomorphic to the Galois ring constructed in the above proposition. Let the homomorphism $\theta : \mathbb{Z}_{p^k}[x] \to R$ be defined by $\theta(h) = h(\alpha)$. Obviously, $Im\ \theta = \mathbb{Z}_{p^k}[\alpha]$.

First, we show that $\mathbb{Z}_{p^k}[x] = R$, by proving that it is of order greater than or equal to p^{kr}. Suppose $\Sigma_{i=0}^{r-1}\mu_i\lambda^i = \Sigma_{i=0}^{r-1}\nu_i\lambda^i$, with $(\mu_0, \mu_1, \ldots, \mu_{r-1}) \neq (\nu_0, \nu_1, \ldots, \nu_{r-1})$. Then, $\Sigma_{i=0}^{r-1}(\mu_i - \nu_i)\, x^i = 0$. Let p^l with $0 \leq l \leq k$ be the largest integer such that $p^l \mid (\mu_i - \nu_i)$. Then $p^l\left(\Sigma_{i=0}^{r-1}\tau_i\lambda^i\right) = 0$ with $(\tau_0, \tau_1, \ldots, \tau_{r-1}) \neq 0\ mod\ p$. But since $l \leq k$, $\Sigma_{i=0}^{r-1}\tau_i\lambda^i$ is a zero divisor and hence $\Sigma_{i=0}^{r-1}\tau_i x^i$ must be a nonzero polynomial in $\mathbb{Z}_{p^k}[x]$ having $\bar{\lambda}$ as a root, which is a contradiction. Hence $\mid \mathbb{Z}_{p^k}[\alpha] \mid = p^{kr}$ and $R = \mathbb{Z}_{p^k}[\alpha]$, so the homomorphism θ is surjective so that $\mathbb{Z}_{p^k}[x]/ker\theta \cong R$. Certainly, $f \in ker\theta$ therefore $f\mathbb{Z}_{p^k}[x] \subset ker\theta$. We must have equality, since otherwise, $\mid \mathbb{Z}_{p^k}[x]/ker\theta \mid < p^{kr}$. Therefore $R \cong \mathbb{Z}_{p^k}[x]/f\mathbb{Z}_{p^k}[x] = R_0$. □

Lemma 1.7.11 *([25]) Let $R \cong GR(p^{kr},\ p^k)$, with $k \geq 1$ and with maximal ideal $J(R)$. Then if $K_0 = R_0/J(R_0) = R_0/pR_0$, then every element of $J(R)$ is uniquely expressible in the form $\Sigma_{i=1}^{k-1}\alpha_i p^i$ where $\alpha_i \in K_0$.*

PROOF.

Let $k = 1$. Then $R = K_0 \cong GF(p^r)$ and the result readily follows. Now suppose $k > 1$. Since $charR = p^k$, then $\mathbb{Z}_{p^k}[x]$ is a subring of R and therefore $\mathbb{Z}_{p^k}[a] \subseteq R$. Let $r \in R$, then there exists $\alpha \in K_0$ so that $r + J(R) = \alpha + J(R)$ which implies that there exists $u \in J(R)$ such that $r = \alpha + u$. Now by Raghavendran (see [25], Theorem 1.7.1), there exists $\alpha_i \in K_0$ and $i \in \{1, \ldots, k-1\}$ such that $u = \Sigma_{i=1}^{k-1}\alpha_i p^i$. Hence $r = \alpha + \Sigma_{i=1}^{k-1}\alpha_i p^i \in \mathbb{Z}_{p^k}[b]$ so that $R \subseteq \mathbb{Z}_{p^k}[a]$. It follows that $R = \mathbb{Z}_{p^k}[a]$. □

Lemma 1.7.12 *[25] Let $R \cong GR(p^{kr},\ p^k)$. If $r \in R$ annihilates p^i for some $0 < i \leq k$, then $r = p^{k-i}x$ for some $x \in R$.*

PROOF.

If $k = 1$, then $r = x$ so the result is trivial. Now, suppose $k > 1$, and $p^i.r = 0$. Assume $1 \leq i \leq k$, otherwise, there is nothing to prove. Using the previous notations, there exists $\mu_0 \in K_0$, $u \in J(R)$ such that $r = \mu_0 + u$. But $u = \Sigma_{i=1}^{k-1}\mu_i p^i$ for some $\mu_i \in K_0$. Thus, $r = \mu_0 + \Sigma_{i=1}^{k-1}\mu_i p^i = \mu_0 + p\mu_1 + \cdots + p^{k-i-1}\mu_{k-i-1} + p^{k-i}x$ where $x = \mu_{k-i} + \cdots + p^{i-1}\mu_{k-1} \in R$. Therefore, $0 = p^i.r = p^i(\mu_0 + p\mu_1 + \cdots + p^{k-i-1}\mu_{k-i-1})$. But $p^i \neq 0$, since $i < k$, so that $\mu_0 + p\mu_1 + \cdots + p^{k-i-1}\mu_{k-i-1}$ is a zero divisor of R and therefore belongs to $J(R)$. But then, $\mu_0 \in J(R)$ and therefore $\mu_0 = 0$. This implies that $0 = p^{i+1}(\mu_1 + p\mu_2 + \cdots + p^{k-i-2}\mu_{k-i-2})$. Therefore $\mu_1 + p\mu_2 + \cdots + p^{k-i-2}\mu_{k-i-2} \in J(R)$ so that now, $\mu_1 = 0$. Repeating the argument another $(k - i - 2)$ times, we obtain that $\mu_0 = \mu_1 = \cdots = \mu_{k-i-1} = 0$. Hence $r = p^{k-i}x$ □

Remark 1.7.13 *The group of automorphisms of a Galois ring is cyclic, and therefore completely classifies the automorphism groups of the Galois rings.*

1.8 Some results on Completely Primary Finite Rings

Let R be a finite ring with identity $1 \neq 0$ and $Z(R)$ be the Jacobson radical of R. Then R is said to be primary if $R/Z(R)$ is simple and is completely primary if $R/Z(R)$ is a division ring (see Wilson [29]). Moreover, notice that $(Z(R))^i \supset (Z(R))^{i+1}$ for each nonzero $(Z(R))^i$. The quotient field $R/Z(R)$ is called the residue field. The quotient spaces $(Z(R))^i/(Z(R))^{i+1}$ may be regarded as vector spaces over the residue field $R/Z(R)$ via the action defined by $(x + Z(R))(y + (Z(R))^{i+1}) =$

$xy + (Z(R))^{i+1}$ for $x \in Z(R)$ and $y \in (Z(R))^{i+1}$.

The following results are due to Raghavendran [25], who has done a more extensive study on completely primary finite rings.

Theorem 1.8.1 *(See [25], Theorem 2) Let R be a completely primary finite ring and $Z(R)$ be its subset of all the zero divisors including the zero. Then,*

(i) *$Z(R)$ is the unique maximal ideal of R and $R/Z(R) \cong GF(p^r)$, for some prime p and positive integer r,*

(ii) *$|R| = p^{nr}$ and $|Z(R)| = p^{(n-1)r}$ for some prime integer p and positive integers n and r,*

(iii) *$(Z(R))^m = (0), \; m \leq n$,*

(iv) *there exists an element $a \in R$ of multiplicative order $p^r - 1$ such that if $\phi : R \to R/Z(R)$ is the canonical homomorphism, then $\phi(a)$ is a primitive element of $R/Z(R)$ and $F_0 =< a > \cup \{0\}$ forms a complete system of coset representatives of $Z(R)$ in R. Further, if $\lambda, \; \mu \in F_0$ with $\lambda - \mu \in Z(R)$, then $\lambda = \mu$,*

(v) *$char R = p^k$ for some k with $1 \leq k \leq m$,*

(vi) *if $char R = p^m$, then R is commutative.*

Proof.

(i) Suppose $u, \; u', \; u'' \in Z(R)$ and $x \in R$, then obviously $u' \pm u'' \in Z(R)$ and $xu = ux \in Z(R)$ so that $Z(R)$ is an ideal of R. Since every element of R is either a zero divisor or a unit, $R - Z(R)$ consists of units and since any ideal which contains a unit is R itself, then $Z(R)$ is maximal. Furthermore $R/Z(R) \cong GF(p^r)$ since $R/Z(R)$ is a finite field. This follows from the fact that every finite division ring is a finite field.

(ii) For each positive integer j, consider $(Z(R))^j \,/\, (Z(R))^{j+1}$ as an $R/Z(R)$ vector space where scalar multiplication is defined as $(x + Z(R)^{j+1})\,(y + (Z(R))^{j+1}) = xy + (Z(R))^{j+1}$. Hence $\mid (Z(R))^j/(Z(R))^{j+1} \mid = p^{k_j r}$. Since $(Z(R))^m = (0)$,

$|R| = \mid\ R/Z(R) \mid\mid Z(R)/(Z(R))^2 \mid \ldots \mid (Z(R))^{m-2}/(Z(R))^{m-1} \mid\mid (Z(R)^{m-1} \mid$

$= p^{r(1+k_1+\cdots+k_{m-1})}$

$= p^{rn}, \quad n = 1 + k_1 + \cdots + k_{m-1}.$

Clearly, $k_i \geq 1$ and $m \leq n$. Therefore $(Z(R))^n = (0)$ and

$|Z(R)| = |R|/|GF(p^r)| = p^{nr}/p^r = p^{(n-1)r}$.

(iii) Finiteness of R implies that for some $u \in Z(R)$, there exists positive integers i and j with $j \leq i$ so that $x^j = x^i$ and $x^j(x^{i-j} - 1) = 0$. Since x^{i-j} is a unit, it does not belong to $Z(R)$. Then $x^j = 0$ and $Z(R)$ is a nil ideal. Finiteness of R also implies that $Z(R)$ is nilpotent, that is, $(Z(R))^m = (0)$.

(iv) Let $R^* = R - Z(R)$, then the canonical homomorphism $\psi : R \to R/Z(R)$ induces a surjective multiplicative group homomorphism $\theta : R^* \to R/Z(R)$. Since $ker\psi = Z(R)$, then $ker\theta = 1 + Z(R)$ so that $1 + Z(R)$ is a normal subgroup of R^*. Now, suppose $<\alpha>= R/Z(R)$ and $a_0 = \theta^{-1}(\alpha)$, then the order of a_0 is $q(p^r - 1)$ and $|R - Z(R)| = p^{nr} - p^{(n-1)r} = p^{(n-1)r}(p^r - 1)$.

So, the order of a_0 is of the form $p^t(p^r - 1)$. But $a = a_0 p^t$ has multiplicative order $p^r - 1$ and $\theta(a_0 p^t) = \alpha p^t$ which also generates $R/Z(R)$ since p^t and $p^r - 1$ are relatively prime. Furthermore, $\theta(F_0) = R/Z(R)$ and therefore F_0

is a complete set of coset representative of $Z(R)$ in R. The last part of the result follows from this property.

(v) $char\,(R/Z(R)) = p$ since $R/Z(R)$ is a finite field. So $p \in Z(R)$ and $p^m = 0$ because $Z(R)$ is nilpotent. Then $charR = p^k$ for $1 \leq k \leq m$.

(vi) Consider the field $K_0 = \{(0, \alpha^t : t = 1, \ldots, p^{r-1})\}$ of p^r elements from (iv), $\alpha - \beta \in Z(R)$ implies that $\alpha = \beta$ for $\alpha,\ \beta \in K_0$. Let $charR = p^m$, then by induction on t, we can show that for $\alpha_t,\ \beta_t \in K_0$ the equation

$$\Sigma_{t=0}^{m-1} p^t.\alpha_t = \Sigma_{t=0}^{m-1} p^t.\beta_t. \tag{1.1}$$

imply that $p^{m-1}\,(\alpha_t - \beta_t) = 0$ and that $\alpha_t - \beta_t = 0$ for $t = 0, \ldots, m-1$. So the set $\{\Sigma_{t=0}^{m-1} p^t.\alpha_t : \alpha_t \in K_0\}$ contains p^{mr} distinct elements and is contained in R, hence $R = \{\Sigma_{t=0}^{m-1} p^t.\alpha_t : \alpha_t \in K_0\}$ and every element of R is uniquely expressed as $\Sigma_{t=0}^{m-1} p^t.\alpha_t,\ \alpha_t \in K_0$, so that R is a commutative ring.

□

Remark 1.8.2 *The ring R is said to be of maximal prime power characteristic if Theorem 1.8.1 part (vi) holds.*

Corollary 1.8.3 *Let R be a completely primary finite ring. Then every element of R is uniquely expressible in the form $\alpha + x,\ \alpha \in K_0 =< a > \cup\{0\}$ and $x \in Z(R)$.*

Remark 1.8.4 *If $k = n$, then $R = \mathbb{Z}_{p^k}[a]$ where a is an element of R of multiplicative order $p^r - 1,\ Z(R) = pR$ and $Aut(R) \cong Aut\,(R/Z(R))$. As earlier stated, R is a Galois ring $GR\left(p^{kr},\ p^k\right)$.*

In addition to the above results, the following theorem due to Wirt [30] was also of great value in this study;

Theorem 1.8.5 *Let R be a completely primary finite ring and $Z(R)$ be its subset of the zero divisors, $|R/Z(R)| = p^r$ and $charR = p^k$. Then R has a coefficient subring R_0 of the form $GR\left(p^{kr},\ p^k\right)$ which is a maximal Galois subring of R. Moreover, there exist $u_1, \ldots, u_h \in Z(R)$ and $\theta_1, \ldots, \theta_h \in Aut(R_0)$ such that*
$R = R_0 \oplus \Sigma_{i=1}^{h} \oplus R_0 u_i$ (as R_0-modules, and $u_i x = x^{\theta_i} u_i$, for every $x \in R_0$ and every $i = 1, \ldots, h$).

The following results are immediate from the above theorem:

(i) $\theta_1, \ldots, \theta_h$ are uniquely determined by R and R_0 (See Theorem 1, [4]).

(ii) θ_i is called the automorphism associated with u_i and $\theta_1, \ldots, \theta_h$ are the associated automorphisms of R with respect to R_0 (See Theorem 8, [25]).

(iii) If R' is another coefficient subring of R, then there exists an invertible element $y \in R$ such that $R' = yR_0y^{-1}$ (See Theorem 8, [25]).

(iv) It is clear that R is commutative if and only if θ_i is an identity automorphism.

1.9 Basic Concepts On Zero Divisor Graphs Of Finite Rings

The following definitions and concepts in graph theory, which can also be found in ([18]and [21]), have been used throughout the thesis.

For the purpose of our study, the graphs of interest are simple non loop, non-directed graphs.

Definition 1.9.1 *A (Simple) graph* $\Gamma = (V, E)$ *is a set V, called the vertex set, and a set of irreflexive, symmetric relations E, on V, called the edge set. If x and y are distinct vertices of* Γ, *then if x and y are related in E, we call the relation an edge between x and y, denoted by* $x - y$ *or* $\{x, y\}$. *The size or order of a graph denoted by* $|V(\Gamma)|$ *is the number of vertices in the vertex set* $V(\Gamma)$. *A subgraph* Γ' *of a graph* Γ *is a graph whose set of vertices and set of edges are all subsets of* Γ. *That is if* $\Gamma' = (V', E') \subset \Gamma = (V, E)$, *then* $V' \subset V$ *and* $E' \subset E$.

Definition 1.9.2 *Let R be a commutative with the identity* $1 \neq 0$ *and* $Z(R)$ *be its subset of zero divisors. We associate with R, a zero divisor graph, denoted by* $\Gamma(R)$ *which is a simple graph with vertex set being the set of non-zero zero-divisors of* R, $Z(R)^{\star} = Z(R) - \{0\}$, *and with* $x - y$ *an edge if and only if* $x \neq y$ *and* $xy = 0$.

Definition 1.9.3 *Chromatic or colouring number of the graph* $\Gamma(R)$ *denoted by* $\chi(R)$ *is the least number of colours which can be assigned to the vertices of a graph so that no adjacent vertices have the same colour.*

Definition 1.9.4 *The degree* $d(v)$ *of a vertex* v *is the number of edges* $E(v)$ *that are incident with or connecting a vertex v. The sum of the degrees of all the vertices is called the degree of the graph.*

In Figure 1.1 below, the degree of vertex a is 3 while the rest of the vertices in the graph are of degree 1 each. Thus $d(a) = 3;\ d(b) = d(c) = d(d) = 1$.

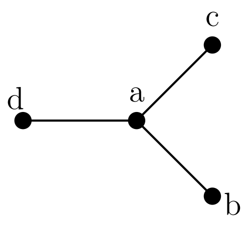


Figure 1.1: Illustration of degree of a vertex of a graph

Definition 1.9.5 *(Path) A path is a sequence of distinct consecutive edges in a graph. The length of a path is the number of edges traversed.*

For instance in Figure 1.2 below, $d \to e \to b \to c \to d$ is a path of length 4.

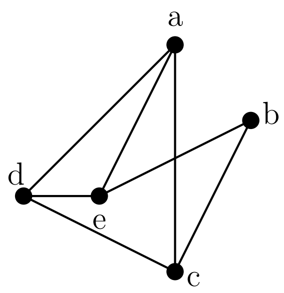


Figure 1.2: Illustration of a path in a graph

Definition 1.9.6 *Let $P = x_0 \cdots x_{k-1}$ be a path with $k \geq 3$. The graph $C := P + x_{k-1}x_0$ is called a cycle. The length of a cycle is its number of edges (or vertices) on the cycle. A cycle of length k is denoted C^k and called a $k-cycle$. The minimum length of a cycle in a graph Γ is called the girth of Γ and is denoted by $gr(\Gamma(R))$. If $\Gamma(R)$ does not contain a cycle, we set $gr(\Gamma(R)) = \infty$.*

Definition 1.9.7 *(Complete graph) A graph $\Gamma(R)$ is called complete if for every $u,\ v \in\ V(\Gamma(R)), u \neq v$, there exists an edge $\{u,\ \ v\}$. Thus, a complete graph with n vertices, denoted by K_n is a graph in which each vertex is connected to each and* *every other vertex by an edge. A non empty graph $\Gamma(R)$ is called connected if there is a path linking any two of its vertices.*

Examples of complete graphs of vertices 1, 2, 3, 4 and 8 respectively are shown in Figure 1.3 below.

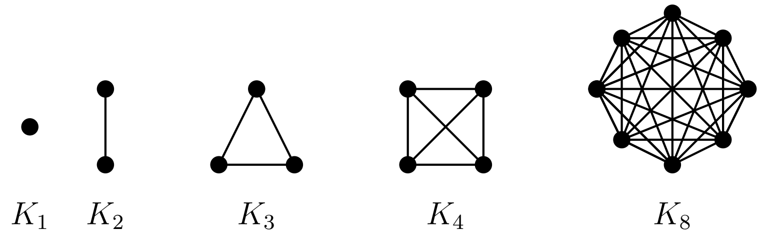


Figure 1.3: Illustration of complete graphs

Definition 1.9.8 *(k-partite verties). Let $Z(R)^*$ denote the set of all nonzero zero divisors of a ring R. Then the vertex set $V = \{V_1, V_2, \dots, V_k\} \subseteq Z(R)^*$ are $k-$ partite of the set V if and only if*

(i) $V_i \neq \emptyset, \ \forall \ 1 \leq i \leq k,$

(ii) $V_i \cap V_j = \emptyset, \ 1 \leq i, j \leq k,$

(iii) $\cup_{i=1}^{k} V_i = V.$

Definition 1.9.9 *A k-partite graph is a graph whose vertices can be partitioned into k-disjoint sets such that no two vertices within the same set are adjacent. If $k = 2$, the graph is a bipartite graph. A k-partite is called complete if every pair of vertices in the k-set are adjacent.*

Definition 1.9.10 *A bipartite graph is a graph whose vertices can be partitioned into two disjoint subsets U and V such that each edge connects a vertex in U to* *one in V and no edge exists between the vertices in the same subset. The bipartite graph is complete if every vertex in U is connected to every vertex in V. If U has n elements and V has m elements, then the complete bipartite graph is denoted by* $K_{m,n}$.

Complete bipartite graphs, $K_{2,3}$ *and* $K_{3,3}$ *are illustrated in Figure 1.4 below.*

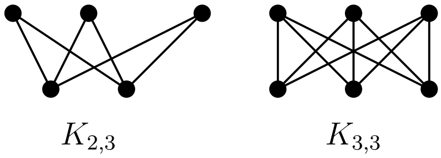


Figure 1.4: Illustration of complete bipartite graphs

Definition 1.9.11 *A star graph* S^r *is the complete bipartite graph* $K_{1,r}$ *in which one of the vertex sets is a singleton.*

The following results apply to star graphs:

(i) If $\Gamma(R)$ is a star graph having p vertices where p is a prime number, then the center of the star graph is idempotent element in R.

(ii) If $\Gamma(R)$ is a star graph of order p and center vertex a, then $\{0, a\}$ forms an ideal in R.

(iii) If $R = \mathbb{Z}_{pq}$ where p and q are distinct prime numbers, then the set of all vertices having the same degree form ideals in R.

(iv) Let $R = \mathbb{Z}_n$, then $n = 2p$ if and only if $\Gamma(R)$ is a star graph of order p.

(v) If $\Gamma(R)$ is a star graph with order p where $p \geq 3$ is prime, then R is a ring such that for all $a \in R$, there exists a $b \in R$ satisfying $a = aba$, also called Von Neumann regular ring.

(vi) If $R = \mathbb{Z}_{p^2}$, then $\mid \Gamma(R) \mid = p - 1$.

Definition 1.9.12 *(Clique and Clique Number) A subgraph of a graph is any subset of vertices together with any subset of edges containing those vertices. An induced subgraph is a subgraph maximal with respect to the number of edges. A complete induced sub-graph K_n of any graph Γ is called a Clique and the Clique number of Γ, denoted by $\omega(\Gamma)$ is the greatest integer $r \geq 1$, such that $K_r \subset \Gamma$.*

Definition 1.9.13 *(Diameter of a graph) For vertices u and v in $\Gamma(R)$, the distance between u and v denoted by $d(u, v)$ is the length of the shortest path from u to v in $\Gamma(R)$, for instance $d(u, u) = 0$ and $d(u, v) = \infty$ if no such path exists. The diameter of $\Gamma(R)$, denoted by $diam(\Gamma(R))$, is defined as $diam(\Gamma(R)) = sup\{d(u, v)|u$ and v are vertices of $\Gamma(R)\}$.*

Definition 1.9.14 *A vertex that has no incident edges is called an isolated vertex.*

Definition 1.9.15 *A graph is said to be almost connected if there exists a path between two non isolated vertices in the graph.*

Definition 1.9.16 *A graph is called singleton if it is a connected graph with zero diameter.*

Theorem 1.9.17 *(see [8], Theorem 2.2) Let R be a commutative ring. Then $\Gamma(R)$ is finite if and only if either R is finite or R is an integral domain. In particular, if $1 \leq |\Gamma(R)| < \infty$, then R is finite with $|R| \leq |Z(R)|^2$ and R is not a field.*

Theorem 1.9.18 *(See [7], Theorem 2.3)Let R be a commutative ring. Then* $\Gamma(R)$ *is connected and* $diam(\Gamma(R)) \leq 3$. *Moreover if* $\Gamma(R)$ *contains a cycle, then* $gr(\Gamma(R)) \leq 7$.

Theorem 1.9.19 *(See [7], Theorem.2.8) Let R be a commutative ring. Then* $\Gamma(R)$ *is complete if and only if either* $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $xy = 0$ *for all* $x, y \in Z(R)$.

Definition 1.9.20 *Let R be a finite ring. The binding number of a graph* $\Gamma(R)$ *denoted by* $b(\Gamma(R))$ *is defined by* $b(\Gamma(R)) = \frac{\mid N(S) \mid}{\mid S \mid}$ *where* $S \subseteq V(\Gamma(R)), S \neq \emptyset$, $N(S) \neq V(\Gamma(R))$ *and satisfy the following conditions*

(i) $N(S) \cup S = V(\Gamma(R))$.

(ii) $N(S) \cap S = \emptyset$.

(iii) the degree, $d(u) \leq d(v)$ *for all* $u \in S, v \in N(S)$.

(iv) no two vertices in S are adjacent.

1.10 Structure of the thesis

The first chapter introduces the basic concepts and definitions which have been used in the thesis. The second chapter accounts for the literature related to this research. The results on the zero divisor graphs of the Galois rings are provided in Chapter 3 while in Chapter 4 we state a well known construction of a class of completely primary finite rings and discuss the results of their zero divisor graphs. In Chapter 5, we have provided results of the zero divisor graphs of completely primary finite rings in which the product of any two zero divisors lies in the Galois subring. Chapter 6 concludes the thesis and provides some recommendations.

Chapter 2

Literature Review

The concept of a zero divisor graph was first introduced by Beck [11]. His graph consists of the vertex set of all elements of the ring R, such that distinct vertices u and v are adjacent if and only if $uv = 0$. This is a simple (no loops) connected graph whose diameter is less than or equal to 2 since the zero vertex is adjacent to every other element of the ring. Beck was mainly interested in the chromatic number of the graph of R and conjectured that the chromatic number of the graph is equal to its clique number. He classified all finite rings with chromatic number strictly less than 4.

Later, Anderson and Naseer [5] proved a counterexample to Beck's conjecture by providing a finite ring whose zero divisor graph had the clique number strictly less than its chromatic number. They proved several results for which the conjecture holds and extended Beck's classification of finite rings with small chromatic number to those cases when the chromatic number is exactly 4.

Anderson and Livingstone [7] simplified Beck's zero divisor graph. The vertex set in their graph consisted of nonzero zero divisors of the ring R and denoted this simple undirected graph by $\Gamma(R)$. This better illustrated the structure of the zero divisors of the ring. They established that if R is a commutative ring, then $\Gamma(R)$ is connected with the diameter, $diam(\Gamma(R)) \leq 3$. They succeeded in showing that $\Gamma(R)$ is finite if and only if the ring R is finite. This approach enabled Anderson and Livingstone [7] to classify finite rings whose graph is complete or is a star graph.

Anderson *et al.* [8] studied the clique number of $\Gamma(R)$ and the relationship between graph isomorphisms and ring isomorphisms. They proved that $\Gamma(R)$ is complete if and only if either $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all non zero x and y in $Z(R)^\star$. In particular they established a fundamental result that if S and T are finite reduced rings which are not fields, then $\Gamma(S)$ and $\Gamma(T)$ are graph isomorphic if and only if S and T are ring isomorphic. They further determined all positive integers n for which $\Gamma(\mathbb{Z}_n)$ is planar, and posed an open problem as to which of the finite rings in general would determine a planar zero divisor graph. This problem was partially answered by Akbari *et al*. [1] where the authors refined the question to local rings whose cardinality is at least 32.

Concurrently, Smith [27], independently provided a complete solution, classifying all the rings with planar zero divisor graphs by listing 44 isomorphism classes altogether. Crucial to all proofs concerning planar graphs, is Kuratowski's Theorem which states that a graph is planar if and only if it contains no subgraph homeomorphic to the complete graph K_5 or the complete bipartite $K_{3,3}$.

Akbari *et al.* [1] listed all the rings that determine a complete r-partite graph. These results are similar to those obtained by Anderson and Livingstone [7], where the ring R is such that $\Gamma(R)$ is a star graph.

Using the zero divisor graph as introduced by Anderson and Livingstone [7], Duane [19] explored p-partite structure of $\Gamma(\mathbb{Z}_n)$ and determined a complete classification of chromatic number of $\Gamma(\mathbb{Z}_n)$ and further, determined how these concepts relate to the prime factorization of n. Duane [19] proved that for each prime integer p,

(i) $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph K_{p-1}.

(ii) $\Gamma(\mathbb{Z}_{p^3})$ is a complete $p-$partite.

(iii) $\Gamma(\mathbb{Z}_{p^k})$, where $k \in \mathbb{Z}^+$, has an induced complete $p-$ partite subgraph.

Nazar *et al.* [23] also investigated the zero divisor graph $\Gamma(R)$ of certain finite rings and were able to characterize the complete bipartite zero divisor graphs of certain finite commutative rings. They characterized $\Gamma(R)$ for $R = \mathbb{Z}_{p^n q}$, where p and q are distinct prime integers while n is a positive integer and proved that if p and q are distinct prime integers and $k > 1$ a positive integer, then the clique number,

$$\omega\left(\Gamma(\mathbb{Z}_{p^k q})\right) = \begin{cases} p^{\frac{k}{2}}, & \text{if } k \text{ is even ;} \\ p^{\frac{k-1}{2}} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Worthy to note is that concurrently, Sankeetha *et al.* [26] were also able to evaluate the binding number of the zero divisor graphs of the ring of integers modulo n. They computed the binding numbers of $\Gamma(\mathbb{Z}_n)$ and proved the following results:-

For each distinct prime integers p, q and $k \in \mathbb{Z}^+$,

(i) $b(\Gamma(\mathbb{Z}_{2p})) = \frac{1}{p-1}$.

(ii) $b(\Gamma(\mathbb{Z}_{p^2})) = \frac{1}{p-2}$.

(iii) $b(\Gamma(\mathbb{Z}_{pq})) = \frac{p-1}{q-1}$, where $p < q$.

(iv) $b(\Gamma(\mathbb{Z}_{2^k})) = \begin{cases} \frac{2^{k-1}-2^{\frac{k}{2}}\Sigma_{i=0}^{\frac{k-4}{2}} 2^i-2}{2^{\frac{k}{2}}\Sigma_{i=0}^{\frac{k-4}{2}} 2^i+1}, & \text{if } k \text{ is even ;} \\ \frac{2^{\frac{k-1}{2}}\left(2^{\frac{k-1}{2}}-\Sigma_{i=0}^{\frac{k-3}{2}} 2^i\right)-1}{2^{\frac{k-1}{2}}\Sigma_{i=0}^{\frac{k-3}{2}} 2^i}, & \text{if } k \text{ is odd.} \end{cases}$

(v) $b(\Gamma(\mathbb{Z}_{4p})) = \frac{3}{2(p-1)}$, $p > 4$.

(vi) $b(\Gamma(\mathbb{Z}_{8p})) = \frac{7}{4(p-1)}$, $p > 8$.

(vii) $b(\Gamma(\mathbb{Z}_{2^kp})) = \frac{2^k-1}{2^{k-1}(p-1)}$, $p > 2^k$.

(viii) $b(\Gamma(\mathbb{Z}_{3^k})) = \frac{7}{3^{k-1}-8}$, $k > 3$.

In this thesis, more generalized results on the binding numbers, the clique numbers and the partiteness of the zero divisor graphs of finite rings of prime power characteristic have been obtained.

The girth is one of the graph invariant properties studied for zero divisor graphs. Known in literature is that if $\Gamma(R)$ has a finite girth then $gr(\Gamma(R)) \leq 4$. If R is *Noetherian* and $\Gamma(R)$ has finite girth, then $gr(\Gamma(R)) \leq 3$.

Anderson and Livingston [7] investigated the interplay between graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of R and showed that if R is *Artinian* ring and $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 4$. The authors conjectured that this upper bound would hold in general and was later confirmed in two independent studies by De Meyer and Schneider [17] and Mulay [22].

Anderson, Axtell and Stickles [6] assessed the preservation of the diameter and girth of the graph of a commutative ring under extensions to polynomials and the power series rings. They investigated the preservation of the diameter and girth under idealizations of commutative rings. They characterized the girth of the zero divisor graph of an idealization and completely established the conditions under which the zero divisor graph of an idealization will be complete. They further provided some conditions for which the zero divisor graph of idealization will have a $diam(\Gamma(R)) = 2$.

In the sequel we consider some important results proved by Anderson *et al.* [6]. They showed that if R is a ring and U is an $R-$ module then,

(a) The girth, $gr\,(\Gamma\,(R \oplus U)) = 3$ if and only if one of the following exists,

(i) $\mid U \mid \geq 4$,

(ii) $U \cong \mathbb{Z}_3$, and one of the following hold.

- there exists a nonzero $r \in R$ such that $r^2 = 0$ or
- there exist distinct $a, b \in Z(R)^{\star}$ such that $ab = 0 = aU = bU$.

(b) The girth, $gr\,(\Gamma\,(R \oplus U)) = \infty$ if and only if one of the following hold.

(i) $U \cong \mathbb{Z}_3$ and $ann\,(U) = 0$ and $R \cong \mathbb{Z}_3$ or

(ii) $U \cong \mathbb{Z}_2$ and either $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or R is an integral domain.

The only case in which $gr\,(\Gamma\,(R \oplus U)) = 4$ is when $U \cong \mathbb{Z}_2$ and R does not meet any of the conditions. For instance, $gr\,(\Gamma\,(R \oplus \mathbb{Z}_2)) = 4$ when $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$.

The diameter of the zero divisor graph of an idealization need not be preserved. Moreover if $diam\,(\Gamma\,(R)) > 1$, then $diam\,(\Gamma\,(R \oplus U)) > 1$. It is worthy to note that a ring R may be such that $\Gamma\,(R)$ is complete and $diam\,(\Gamma\,(R \oplus U)) > 1$ besides other possible combination between the diameter of R and that of $R \oplus U$. The necessary and sufficient conditions to guarantee that $\Gamma\,(R \oplus U)$ is complete have been provided and some results for the cases when $diam\,(\Gamma\,(R \oplus U)) = 2$ have also been established.

The authors [6] proved that if $\Gamma\,(R) \neq \emptyset$, then $\Gamma\,(R \oplus U)$ is complete if and only if

$R \oplus U$ satisfies the following properties;

(i) $(Z(R))^2 = 0$.

(ii) For every element $r \in R, ru \neq 0$ for all $u \in U^\star = U \backslash \{0\}$.

(iii) If $r \in Z(R)^\star$, then $rU = 0$.

They established that $\Gamma(R \oplus U)$ is complete if and only if $Z((R \oplus U))^2 = 0$ and even though it is clear that $\Gamma(R \oplus U) > 1$ if $\Gamma(R) > 1$, and that idealizations need not preserve the diameter of the zero divisor graph, it is possible to construct idealizations which preserve the diameter of the zero divisor graphs. For instance, $diam(\Gamma(\mathbb{Z}_8)) = 2$ and $diam(\Gamma(\mathbb{Z}_8 \oplus \mathbb{Z}_2)) = 2$, (see [6]).

The classification of the diameter of an idealization would be exhausted if it was possible to find the necessary and sufficient conditions for ensuring that $diam(\Gamma(R \oplus U)) = 2$. This classification is still open even though it has been noted that the characterization of the diameter 3 is manageable.

The authors [6] also found that $diam(\Gamma(R \oplus U)) = 2$ if $R \oplus U$ is such that $(Z(R))^2 = 0$ and for every element $r \in R$, $ru \neq 0$ for all $u \in U^\star = U \backslash \{0\}$ but if $r \in Z(R)^\star$, then $rU \neq 0$. Alternatively, $diam(\Gamma(R \oplus U)) = 2$ if $(Z(R))^2 \neq 0$ and for every element $r \in R$, $ru \neq 0$ for all $u \in U^\star = U \backslash \{0\}$, but if $r \in Z(R)^\star$, then $rU = 0$.

Mulay [22] introduced another version of the zero divisor graph associated to a ring R. He considered two zero divisors u, $v \in Z(R)^\star$ to be equivalent if $ann_R(u) = ann_R(v)$. His graph, denoted by $\Gamma_E(R)$, is a simple graph with the vertex set equal to the set of equivalence classes $\{[u] \mid u \in Z(R)^\star\}$ so that distinct equivalence classes

$[u]$ and $[v]$ are adjacent in $\Gamma_E(R)$ if and only if $uv = 0$ in R. Mulay [22] showed that for a given a ring R the graph is connected with $diam(\Gamma_E(R)) \leq 3$.

Spiroff and Wickam [28] compared and contrasted $\Gamma_E(R)$ with $\Gamma(R)$. A critical distinction between $\Gamma_E(R)$ and two earlier versions of zero divisor graphs is that $\Gamma_E(R)$ can be finite even when R is infinite, which therefore gives a more explicit visual description of the zero divisor structure of the ring. For instance they showed that if $R = \mathbb{Z}/(3) \times \mathbb{Z}/(3)$, then $\Gamma(R)$ is a 4 cycle graph with $gr(\Gamma(R)) = 4$ while $\Gamma_E(R)$ is an edge. Some of their other findings were that if R is *Noetherian* ring, then $\Gamma_E(R)$ is complete K_2 graph and if $\Gamma_E(R)$ is complete bipartite, $K_{n,m}$, then $n = 1$ so that $\Gamma_E(R)$ is a star graph. They also found that if $\Gamma_E(R)$ has at least 3 vertices then it is not a cycle or more generally not regular. One other significant aspect of Mulay's graph is that its vertices correspond to the annilator ideals in the ring R so that the associated primes of R are represented in $\Gamma_E(R)$.

These results have proved to be useful for comparison reasons to the results that have been obtained for the zero divisor graphs of the finite rings constructions investigated in this thesis. In most research articles related to our study, it has been argued that the zero divisor graph in which the vertices are nonzero zero divisors yield better characterization of the zero divisors of commutative rings. This study has extended the idea of the zero divisor graphs of idealizations by providing constructions of two classes of more generalized idealizations and investigated the structures of their zero divisors.

Taking into consideration the fact that any finite ring is decomposable into a finite direct sum of completely primary finite rings, this study has characterized the zero divisor graphs based on their invariant geometrical properties and has made an immense contribution towards the classification of these finite rings.

Chapter 3

Zero divisor graphs of Galois Rings

Throughout this section, R_0 shall denote a Galois ring. The objective of this section is to investigate the properties of the zero divisor graphs of Galois ring $R_0 = GR(p^{kr},\ p^k)$ of order p^{kr} and characteristic p^k where p is prime and k and r are positive integers.

Let R_0 be Galois ring and $Z(R_0)$ be its subset of the zero divisors (including zero). Then $Z(R_0)$ is a unique maximal ideal and is hence the Jacobson radical of R_0. Associate with R_0, the graph $\Gamma(R_0)$ whose vertices are the elements of the set $Z(R_0)^{\star}$. Two distinct vertices $x, y \in Z(R_0)^{\star}$ are adjacent if $xy = 0$. We also explore the graph of equivalent vertices in R_0 denoted by $\Gamma_E(R_0)$ which was introduced by Mulay in [9] as a simple graph with vertex set $Z(R_0)^{\star}/\sim$, such that $[x], [y] \in Z(R_0)^{\star}/\sim$ are adjacent if the product $xy = 0$. Consider $x \in Z(R_0)^{\star}$ and $s \neq 1$ a unit element in R_0. Let the vertices x and sx in $\Gamma(R_0)$ be distinct and $[x] = [sx]$ in $\Gamma_E(R_0)$, (Note that $[x] = [sx]$ implies that $ann(x) = ann(sx)$). We investigate the connectedness of $\Gamma(R_0)$ and $\Gamma_E(R_0)$ and also compute diameter, girth and binding number as well as clique number of both $\Gamma(R_0)$ and $\Gamma_E(R_0)$. We begin with the trivial case and later consider the general case.

3.1 Trivial Case

Lemma 3.1.1 *Let $R_0 = GR(p^{kr},\ p^k)$. Consider the graph $\Gamma(R_0)$ whose vertices are given by $p^{lr}\alpha$ with $\alpha \in R_0^\star$ and l, a positive integer.*

Then

$$deg(p^{lr}\alpha) = \begin{cases} p^{lr} - 1, & if\ \ 2l < k; \\ p^{lr} - 2, & if\ \ 2l \geq k. \end{cases}$$

PROOF.

We have two cases to consider.

Case (i): When $r = 1$. So $GR(p^{kr},\ p^k) = G(p^k,\ p^k) \cong \mathbb{Z}_{p^k}$.

Clearly $R_0 = \mathbb{Z}_{p^k}$ and $\gcd(\alpha, p^k) = 1$. So all the vertices adjacent to $(p^l\alpha)$ in the set $p\mathbb{Z}_{p^k}$ of vertices of the graph $\Gamma(R_0)$ are the same vertices adjacent to (p^l). Hence to find the $deg(p^l\alpha)$ it suffices to find all vertices adjacent to (p^l). Let n be the number of vertices adjacent to (p^l) in $\Gamma(R_0)$. Now, the first term in this sequence of vertices is p^{k-l} and the n^{th} term is $p^{k-l} + (n-1)p^{k-l}$. Since the last term in the sequence is $p^k - p^{k-l}$, it easily follows that $p^{k-l} + (n-1)p^{k-l} = p^k - p^{k-l}$ leading to $n = p^l - 1$ if $p^{k-l} > p^l$ or $k > 2l$. If $p^{k-l} \leq p^l$ or $k \leq 2l$, then $p^{2l-k}(p^{k-l}) = p^l$ is adjacent to itself, so that $deg(p^l) = p^l - 2$.

Case (ii): When $r > 1$, then the degree of $f(x) > 1$. So if $p^l\alpha \in R_0$ where $\alpha \in R_0^\star$, then $deg(p^l\alpha) = |ann(p^l\alpha) - \{0, p^l\alpha\}|$. Now $ann(p^l\alpha) = p^{(k-l)}R_0$. Hence

$$|ann(p^l\alpha)| = |p^{k-l}R_0| = p^{(k-(k-l))r} = p^{lr}.$$

When $k > 2l$, we claim that $p^l\alpha \notin ann(p^l\alpha)$. Without loss of generality, let $k = 2l + 1$, then $p^{(k-l)}\alpha = p^{(2l+1-l)}\alpha = p^{(l+1)}\alpha$. This implies that $p^{(l+1)}\alpha$ is the least element in $ann(p^l\alpha)$. When $k \leq 2l$, we claim that $p^l\alpha \in ann(p^l\alpha)$. Without loss of generality, let $k = 2l - 1$, then $p^{(k-l)}\alpha = p^{(2l-1-l)}\alpha = p^{(l-1)}\alpha$ is the least element in $ann(p^l\alpha)$. Therefore conclusively, when $k > 2l$, then $|ann(p^l\alpha) - \{0, p^l\alpha\}| = p^{lr} - 1$ and when $k < 2l$, $|ann(p^l\alpha) - \{0, p^l\alpha\}| = p^{lr} - 2$. which completes the proof. □

Proposition 3.1.2 *Let $R_0 = GR(p^k,\ p^k)$ be the Galois ring of order p^k and characteristic p^k. Then the graph of R_0 is*

$$\Gamma(R_0) = \begin{cases} p^{\frac{k}{2}} - 1 \ - partite, & \textit{if } k \textit{ is even;} \\ p^{\frac{k-1}{2}} \ - partite, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof.

Let $R_0 = GR(p^k,\ p^k)$ and consider the set $Z(R_0)^\star = Z(R_0)\backslash\{0\} = pR_0\backslash\{0\}$ of all the non zero zero divisors of R_0. We partition $Z(R_0)^\star$ as follows;

Case (i): k is an even integer.

Partition $Z(R_0)^\star$ into the following subsets.

$V_1 = Z(R_0)^\star\backslash\{\bigcup\{j(p^{\frac{k}{2}})\},\ \ 2 \leq j \leq p^{\frac{k}{2}} - 1\}$ and $V_j = \{j(p^{\frac{k}{2}})\},\ \ 2 \leq j \leq p^{\frac{k}{2}} - 1$. Clearly, each of the V_i for $1 \leq i \leq p^{\frac{k}{2}} - 1$ are distinct non empty sets, containing non adjacent vertices. Thus; $V_i \neq \emptyset$ for all $1 \leq i \leq p^{\frac{k}{2}} - 1$, $V_1 \cap V_j = \emptyset$ for all $2 \leq j \leq p^{\frac{k}{2}} - 1$ and $V_j \cap V_l = \emptyset$ for all $j \neq l$, $2 \leq j,\ l \leq p^{\frac{k}{2}} - 1$. Finally $Z(R_0)^\star = V_1 \cup \cup_{j=2}^{p^{\frac{k}{2}}-1}\{V_j\} = \cup_{i=1}^{p^{\frac{k}{2}}-1}\{V_i\}$. Therefore $\Gamma(R_0)$ is $(p^{\frac{k}{2}} - 1) -$ partite.

Case (ii): k is an odd integer.

Partition $Z(R_0)^\star$ into the following subsets. $V_1 = Z(R_0)^\star \backslash \bigcup\{(j-1)p^{\frac{k+1}{2}}\}$ for $2 \leq j \leq p^{\frac{k-1}{2}}$ and $V_j = \{(j-1)p^{\frac{k+1}{2}}\}$ for all $2 \leq j \leq p^{\frac{k-1}{2}}$. Clearly, each of the V_i for all $1 \leq i \leq p^{\frac{k-1}{2}}$ contains non adjacent vertices. Moreover, $V_i \neq \emptyset$ for $1 \leq i \leq p^{\frac{k-1}{2}}$, $V_1 \cap V_j = \emptyset$ for $1 \leq j \leq p^{\frac{k-1}{2}}$ and $V_j \cap V_l = \emptyset$ for $j \neq l$, $2 \leq j, l \leq p^{\frac{k-1}{2}}$. Finally, $Z(R_0)^\star = V_1 \bigcup \cup_{j=2}^{p^{\frac{k-1}{2}}} \{V_j\} = \bigcup_{i=1}^{p^{\frac{k-1}{2}}} \{V_i\}$. It thus follows that for odd integer k, $\Gamma(R_0)$ is $p^{\frac{k-1}{2}}$ $-$ partite. □

Proposition 3.1.3 *Let $R_0 = GR(p^k,\ p^k), k \geq 3$. Then*

(i) $diam(\Gamma(R_0)) = 2$.

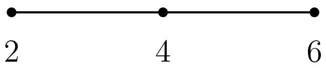
$$(ii)\ gr(\Gamma(R_0)) = \begin{cases} \infty, & \text{if } p = 2 \text{ and } k = 3; \\ 3, & \text{elsewhere.} \end{cases}$$

$$(iii)\ b(\Gamma(R_0)) = \begin{cases} \frac{p^{\frac{k}{2}} - 2}{p^{k-1} - p^{\frac{k}{2}} + 1}, & \text{if } k \text{ is even;} \\ \frac{p^{\frac{k-1}{2}} - 1}{p^{k-1} - p^{\frac{k-1}{2}}}, & \text{if } k \text{ is odd.} \end{cases}$$

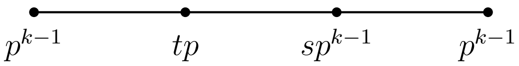
Proof.

To establish (i), the vertex p^{k-1} of $\Gamma(R_0)$ is adjacent to every other vertex. Suppose $t_1 + t_2 \not\equiv 0 \pmod{p^k}$, then the vertices p^{t_1} and p^{t_2} are nonadjacent. Thus $diam\,(\Gamma(R_0)) = 2$.

To prove (ii), let $p = 2$ and $k = 3$, then the graph, $\Gamma(R_0)$ shown below



is a bipartite graph, and therefore does not admit a polygon as a subgraph. Elsewhere, for $t \in \mathbb{Z}^+$ and $2 \leq s \leq p - 1$, the graph



is a triangle. Moreover, p^{k-1} is adjacent to all the vertices. Thus $\Gamma(R_0)$ admits K_3.

To prove (iii), let k be even and let $V_1 = Z(R_0)^\star - \{j(p^{\frac{k}{2}}),\ 2 \leq j \leq p^{\frac{k}{2}} - 1\}$. Then by Definition 1.9.20, $N(V_1) = \{j(p^{\frac{k}{2}}),\ 2 \leq j \leq p^{\frac{k}{2}} - 1\}$. Now we have that $\mid V_1 \mid = (p^{k-1} - 1) - (p^{\frac{k}{2}} - 2) = p^{k-1} - p^{\frac{k}{2}} + 1$ and $\mid N(V_1) \mid = p^{\frac{k}{2}} - 2$ so that the binding number, $b\,(\Gamma(R_0)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{\frac{k}{2}}-2}{p^{k-1}-p^{\frac{k}{2}}+1}$.

When k is odd, we have that $V_1 = Z(R_0)^\star - \{(j-1)(p^{\frac{k+1}{2}}),\ 2 \leq j \leq p^{\frac{k-1}{2}}\}$ and $N(V_1) = \{(j-1)(p^{\frac{k+1}{2}}),\ 2 \leq j \leq p^{\frac{k-1}{2}}\}$. Then, $\mid V_1 \mid = p^{k-1} - p^{\frac{k-1}{2}}$ and $\mid N(V_1) \mid = (p^{k-1} - 1) - (p^{k-1} - p^{\frac{k-1}{2}}) = p^{\frac{k-1}{2}} - 1$. So $b\,(\Gamma(R_0)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{\frac{k-1}{2}}-1}{p^{k-1}-p^{\frac{k-1}{2}}}$.

□

Example 3.1.4 *Let $R_0 = GR(3^4,\ 3^4) = \mathbb{Z}_{81}$. Here, $p = 3$ and $k = 4$. Then*

$Z(R_0)^\star = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78\}$.

Then we have $V_1 = \{3, 6, 9, 12, 15, 21, 24, 30, 33, 39, 42, 48, 51, 57, 60, 66, 69, 75, 78\}$,

$V_2 = \{18\}, V_3 = \{27\}, V_4 = \{36\}, V_5 = \{45\}, V_6 = \{54\}, V_7 = \{63\}$ *and* $V_8 = \{72\}$.

Thus, $\Gamma(\mathbb{Z}_{81})$ is 8-partite or octa-partite as seen in Figure 3.1 below with;

$diam(\Gamma(R_0)) = 2$.

$gr(\Gamma(R_0)) = 3$.

$b(\Gamma(R_0)) = \frac{7}{19}$.

$\omega(\Gamma(R_0)) = 8$.

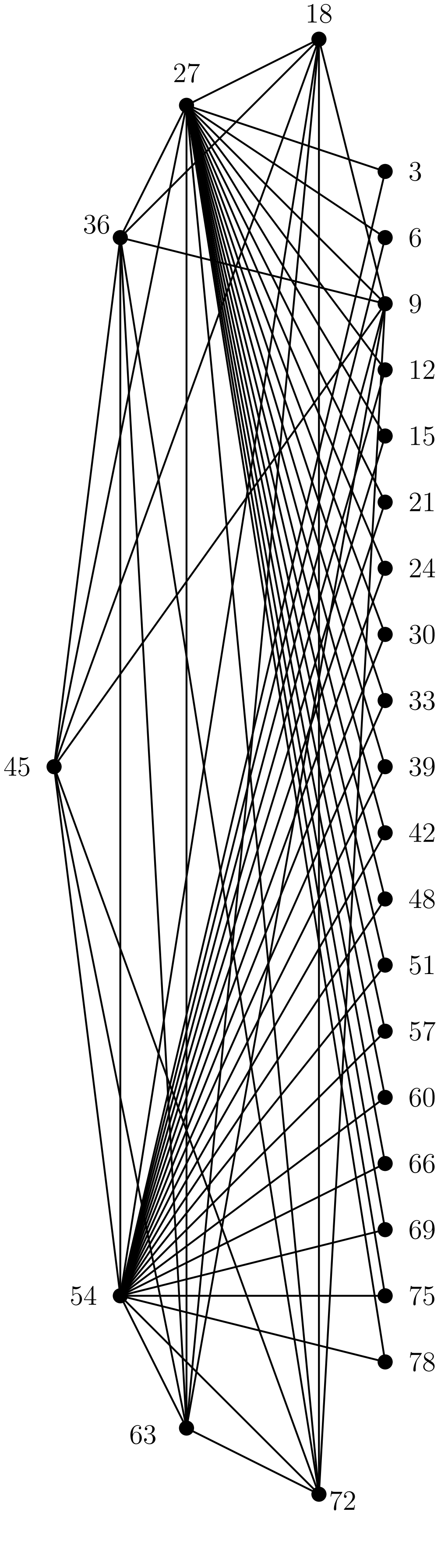


Figure 3.1: The zero divisor graph of $GR(3^4,\ 3^4)$

Example 3.1.5 *Let $R_0 = GR(2^3,\ 2^3) = \mathbb{Z}_8$. Here, $p = 2$ and $k = 3$. Then $Z(R_0)^\star = \{2, 4, 6\}$, with $V_1 = \{2, 4, 6\} \setminus \{4\} = \{2, 6\}$ and $V_2 = \{4\}$.*

So the zero-divisor graph $\Gamma(\mathbb{Z}_8)$ of $\mathbb{Z}_8$ is bi-partite graph drawn in Figure 3.2 below with;

$diam(\Gamma(R_0)) = 2$.

$gr(\Gamma(R_0)) = \infty$.

$b(\Gamma(R_0)) = \frac{1}{2}$.

$\omega(\Gamma(R_0)) = 2$.

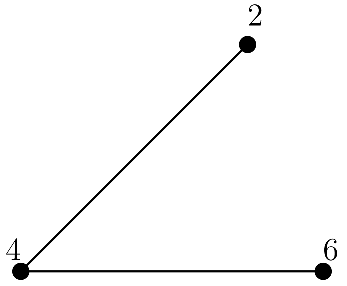


Figure 3.2: The zero divisor graph of $GR(2^3,\ 2^3)$

Remark 3.1.6 *We observe from the above examples that if $R = \mathbb{Z}_{p^k}$ and $\Gamma(R)$ is a perfect graph (i.e. if for every subgraph $H \subseteq \Gamma(R)$, $\omega(H) = \chi(H)$), then the partite number is equal to the clique number of $\Gamma(R)$. So the clique number for $\Gamma(R)$ is $p^{\frac{k}{2}} - 1$ if k is even and is $p^{\frac{k-1}{2}}$ if k is odd.*

Automorphisms of zero divisor graphs of trivial Galois rings

Trivial Galois rings can be exhibited in two ways: when $k = 1, R_0 = GR(p^r,\ p)$ and when $r = 1, R_0 = GR(p^k,\ p^k)$. In the former case, the zero divisor graph is empty. It is therefore of interest to describe the group of automorphisms of the zero divisor graphs of Galois rings in the latter case.

We use the ideas based on the findings of Anderson and Livingston [7]. Distinct ring automorphisms of R induce distinct graph automorphisms of $\Gamma(R)$ provided R is a finite ring and is not a field.

An automorphism f of a graph $\Gamma(R)$ is a bijection $f : \Gamma \longrightarrow \Gamma$ which preserves adjacency. The set $Aut(\Gamma)$ of all graph automorphisms of Γ forms a group under the usual composition of functions. Moreover if $|V(\Gamma)| = p^k$, then $Aut(\Gamma)$ is isomorphic to a subgroup of S_{p^k} and hence it follows that $Aut(K_{p^k}) \cong S_{p^k}$. Infact, for a graph Γ of order p^k, $Aut(\Gamma) \cong S_{p^k}$ if and only if $\Gamma = K_{p^k}$. Now, by restricting each $f \in Aut(R)$ to $Z(R)^\star$ we obtain a natural group homomorphism $\phi : Aut(R) \longrightarrow Aut(\Gamma(R))$. Generally, $Aut(\Gamma(R)) \gg Aut(R)$.

Remark 3.1.7 *An $f \in Aut(R)$ is completely determined by its action on $Z(R)$.*

Theorem 3.1.8 *Let R be a completely primary finite ring which is not a field, and let $f \in Aut(R)$. If $f(x) = x$, for all $x \in Z(R)$, then $f = 1_R$. Thus $\phi : Aut(R) \longrightarrow Aut(\Gamma(R))$ is a monomorphism.*

Now consider the ring $\mathbb{Z}_{p^k}$. For $p^k \geq 4, k \neq 1$. Let
$X = \{d \in \mathbb{Z} \mid 1 < d < p^k \text{ and } d \mid p^k\}$.
For each $d \in X$, let $V_d = \{x \in \mathbb{Z} \mid 1 < x < p^k \text{ and } \gcd(x,\ p^k) = d\}$. We note that $Z(\mathbb{Z}_{p^k})^\star$ is the disjoint union of $V_{d's}$. Furthermore, notice that two vertices have the same degree if and only if they are in the same V_d.

Proposition 3.1.9 *If $k \geq 2$ is an integer, then*

(i) $|Aut(\Gamma(\mathbb{Z}_{2^k}))| \cong \Pi_{i=2}^{k}(2^{k-i})!$ *if* $p = 2$,

(ii) $|Aut(\Gamma(\mathbb{Z}_{p^k}))| \cong \Pi_{i=2}^{k} p^{k-i}(p-1)!$ *if* $p \neq 2$.

Proof.

(i) The set $X = \{2, 4, \ldots, 2^{k-1}\}$, then

$V_2 = \{2t \ : 1 \leq t < 2^{k-1} \text{ is odd } \};$

$V_4 = \{4t \ : 1 \leq t < 2^{k-2} \text{ is odd } \};$

$\vdots$

$V_{2^{k-1}} = \{2^{k-1} \ \}.$

Upon counting, $|V_2| = 2^{k-2}$, $|V_4| = 2^{k-3}$, and continuing inductively in this manner, we have $|V_{2^{k-2}}| = 2$ and $|V_{2^{k-1}}| = 1$. Then $Aut(\Gamma(\mathbb{Z}_{2^k})) \cong \Pi_{i=2}^{k} S_{2^{k-i}}$ and the required result readily follows.

(ii) Taking a prime integer $p \neq 2$, and the set $X = \{p, p^2, \ldots, p^{k-1}\}$, then,

$V_p = \{pt \ : 1 \leq t < p^{k-1} \text{ and } (t, p) = 1\};$

$V_{p^2} = \{p^2 t \ : 1 \leq t < p^{k-2} \text{ and } (t, p^2) = 1\};$

$\vdots$

$V_{p^{k-1}} = \{p^{k-1} t \ : 1 \leq t < p \text{ and } (t, p^{k-1}) = 1\}.$

Upon counting, we have $|V_p| = p^{k-2}(p-1)$, $|V_{p^2}| = p^{k-3}(p-1)$, and continuing inductively in this manner, we have $|V_{p^{k-2}}| = p(p-1)$ and $|V_{p^{k-1}}| = (p-1)$. Then $Aut(\Gamma(\mathbb{Z}_{p^k})) \cong \Pi_{i=2}^{k} S_{p^{k-i}(p-1)}$ and the required result readily follows.

□

3.2 General Case

Theorem 3.2.1 *(See [2]) Let R be a commutative ring (not necessarily Galois). Then $\Gamma(R)$ is finite if and only if R is finite or an integral domain.*

Remark 3.2.2 *By the immediate theorem $\Gamma(R_0)$ is finite since R_0 is finite.*

Lemma 3.2.3 *Let $R_0 = GR(p^r,\ p)$, then $Z(R_0)^\star = \emptyset$.*

Proof.

R_0 has no nonzero zero divisors since it is a field. □

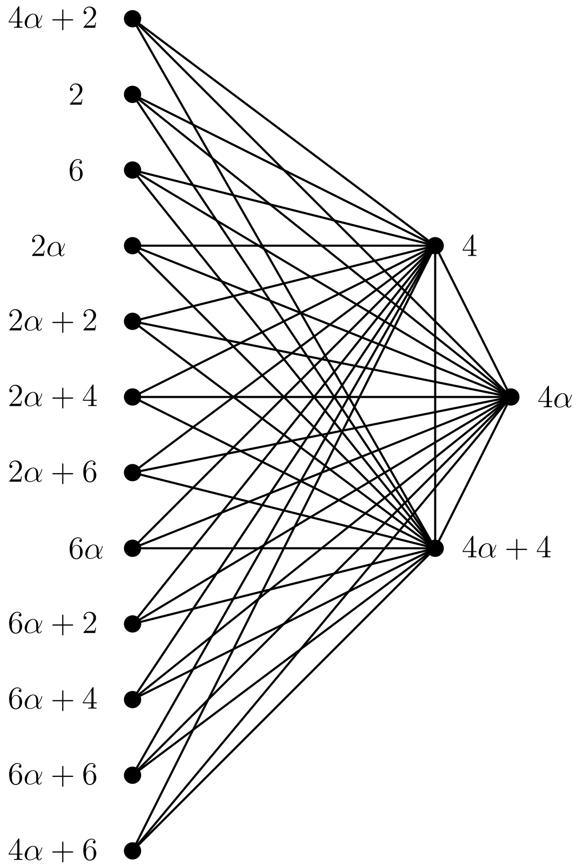
Proposition 3.2.4 *Let $R_0 = GR(p^{2r},\ p^2)$. Then $\Gamma(R_0) = K_{p^r-1}$ and $\Gamma_E(R_0)$ is a single vertex.*

Proof.

Since $(Z(R_0))^2 = 0$, each zero divisor is adjacent to each other. But $|Z(R_0)^\star| = p^r - 1$, so that $\Gamma(R_0)$ is complete on $p^r - 1$ vertices. That $\Gamma_E(R_0)$ is a single vertex follows from the fact that $ann(Z(R_0)) = Z(R_0)$. □

Example 3.2.5 *For $k = 3,\ r = 2$, we have $R_0 = GR(2^6,\ 2^3) = \mathbb{Z}_8[x]/(x^2+1)$. So let α be the root of $f(x) = x^2 + 1$ in $\mathbb{Z}_8$. Then $Z(R_0)^\star = \{2, 4, 6, 2\alpha, 2\alpha+2, 2\alpha+4, 2\alpha+6, 4\alpha, 4\alpha+2, 4\alpha+4, 4\alpha+6, 6\alpha, 6\alpha+2, 6\alpha+4, 6\alpha+6\}$.*

We partition $Z(R_0)^\star$ as follows: $V_1 = \{2, 6, 2\alpha, 2\alpha+2, 2\alpha+4, 2\alpha+6, 4\alpha+2, 4\alpha+6, 6\alpha, 6\alpha+2, 6\alpha+4, 6\alpha+6\}$, $V_2 = \{4\}$, $V_3 = \{4\alpha\}$, $V_4 = \{4\alpha+4\}$. So $\Gamma(R_0)$ is a 4-partite graph as seen in Figure 3.3 below with the following characteristics; $diam(\Gamma(R_0)) = 2$, $gr(\Gamma(R_0)) = 3$ and $b(\Gamma(R_0)) = \frac{1}{4}$ while $\omega(\Gamma(R_0)) = 4$.



4-partite graph

Figure 3.3: The zero divisor graph of $GR(2^6,\ 2^3)$

Note that $V_1 = \{2, 6, 2\alpha, 2\alpha{+}2, 2\alpha{+}4, 2\alpha{+}6, 4\alpha{+}2, 4\alpha{+}6, 6\alpha, 6\alpha{+}2, 6\alpha{+}4, 6\alpha{+}6\}$ represent the equivalence class [2] while V_i , $i \in \{2,\ 3,\ 4\}$, is the equivalence class [4] so that $\Gamma_E(R_0)$ is an edge as shown below in Figure 3.4.

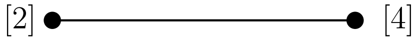


Figure 3.4: The graph Γ_E of zero divisors of $GR(2^6,\ 2^3)$

Remark 3.2.6 *Observe by the above two examples that if* $R_0 = GR(p^{3r},\ p^3)$*, then* $\Gamma(R_0)$ *is more crowded or messy while* $\Gamma_E(R_0)$ *is a single edge.*

Proposition 3.2.7 *Let $R_0 = GR(p^{kr},\ p^k)$ where $k \geq 3, r \in \mathbb{Z}^+$. Then,*

$$\Gamma(R_0) = \begin{cases} (p^{\frac{k}{2}r} - 1) - \ partite, & if\ k\ is\ even; \\ \\ p^{\frac{k-1}{2}r}\ -\ partite, & if\ k\ is\ odd. \end{cases}$$

Proof.

Clearly $Z(R_0)^\star = Z(R_0)\backslash\{0\} = pR_0\backslash\{0\}$. We have two cases to consider.

Case I: When k is an even integer.

Let $\epsilon_1, \ldots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon_1}, \ldots, \bar{\epsilon_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $GR(p)$. We now partition $Z(R_0)^\star$ into the following subsets; $U_i = \{\Sigma a_i\epsilon_i\}$ where $1 \leq i \leq r$ and $a_i \in \{0, j(p^{\frac{k}{2}})\}$ for $1 \leq j \leq p^{\frac{k}{2}} - 1$; $V_{\Sigma a_i\epsilon_i} = U_i\backslash\{0, p^{\frac{k}{2}}\}$ and $V_1 = Z(R_0)^\star\backslash\bigcup_i V_{\Sigma a_i\epsilon_i}$. Now, for each $i = 1, \ldots, r$, $V_{\Sigma a_i\epsilon_i} \neq \emptyset$ and each of the $V_{\Sigma a_i\epsilon_i}$ contains no adjacent vertices. $V_1 \cap V_{\Sigma a_i\epsilon_i} = \emptyset$ and the sets $V_{\Sigma a_i\epsilon_i}$ are all mutually disjoint. Moreover, $Z(R_0) = V_1 \cup \{\underset{r}{\cup} V_{\underset{i}{\Sigma} a_i\epsilon_i}\}$. Thus $\Gamma(R_0)$ is $(p^{\frac{k}{2}r} - 1) - partite.$

Case II: When k is an odd integer.

We partition $Z(R_0)^\star$ into the following subsets. Let $U_i = \{\Sigma a_i\epsilon_i\}$ where $1 \leq i \leq r$ and $a_i \in \{0, (j)(p^{\frac{k+1}{2}})\}$ for $2 \leq j \leq p^{\frac{k-1}{2}}$; $V_{\Sigma a_i\epsilon_i} = U_i\backslash\{0\}$ and $V_1 = Z(R_0)^\star\backslash U_i$. The rest of the proof is similar to Case I above with slight modifications. □

Proposition 3.2.8 *Let $R_0 = GR(p^{kr},\ p^k)$, $k \geq 3$. Then the*

(i) diameter, $diam(\Gamma(R_0)) = 2$.

(ii) girth, $gr(\Gamma(R_0)) = \begin{cases} \infty, & p = 2, k = 3, r = 1; \\ \\ 3, & elsewhere. \end{cases}$

(iii) binding number,

$$b(\Gamma(R_0)) = \begin{cases} \dfrac{p^{\frac{k}{2}r} - 2}{p^{(k-1)r} - p^{(\frac{k}{2})r} + 1}, & \textit{if } k \textit{ is even;} \\ \dfrac{p^{(\frac{k-1}{2})r} - 1}{p^{(k-1)r} - p^{(\frac{k-1}{2})r}}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof.

Since $R_0 = GR(p^{kr}, p^k)$ is of characteristic p^k , proof to (i) and (ii) are similar to Proposition 3.1.3.

(iii) Let $\epsilon_1, \ldots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$, such that $\bar{\epsilon_1}, \ldots, \bar{\epsilon_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield F_p. Let k be even and partition $Z(R_0)^\star$ as done in the proof of Proposition 3.2.7, then by definition of V_1, $N(V_1) = \bigcup_i V_{\Sigma a_i\epsilon_i}$. So $|N(V_1)| = p^{\frac{k}{2}r} - 2$ and $|V_1| = |Z(R_0)| - |\bigcup_i V_{\Sigma a_i\epsilon_i}| = (p^{(k-1)r} - 1) - (p^{\frac{k}{2}r} - 2) = p^{(k-1)r} - p^{\frac{k}{2}r} + 1$. Then $b(\Gamma(R_0)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{\frac{k}{2}r}-2}{p^{(k-1)r}-p^{(\frac{k}{2})r}+1}$ if k is even.

If k is odd, then $|N(V_1)| = |\bigcup_i V_{\Sigma a_i\epsilon_i}| = p^{(\frac{k-1}{2})r} - 1$ and $|V_1| = |Z(R_0)^\star \backslash \bigcup_i V_{\Sigma a_i\epsilon_i}| = |Z(R_0)^\star| - |\bigcup_i V_{\Sigma a_i\epsilon_i}| = (p^{(k-1)r} - 1) - (p^{(\frac{k-1}{2})r} - 1) = p^{(k-1)r} - p^{(\frac{k-1}{2})r}$. Then $b(\Gamma(R_0)) = \frac{|N(V_1)|}{|V_1|} = \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1)r}-p^{(\frac{k-1}{2})r}}$. □

Corollary 3.2.9 *Let $R_0 = GR(p^{kr},\ p^k)$ where $k \geq 4$. Then,*

$$\Gamma_E(R_0) = \begin{cases} \frac{k}{2} - \textit{partite}, & \textit{if } k \textit{ is even;} \\ \frac{k+1}{2} - \textit{partite}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. Case I: When k is even.

The vertex set of $\Gamma_E(R_0)$ is partitioned into the following subsets $V_1 = \{J^l\}$ where $1 \leq l \leq \frac{k}{2}$ and $V_i = \{J^i\}$ with $\frac{k}{2} < i \leq k-1$. For each i, $V_1 \cap V_i = \emptyset$ and V_i are mutually disjoint. Moreover $V_1 \bigcup \cup_{i=\frac{k}{2}}^{k-1}\{V_i\} = V(\Gamma_E(R_0))$ where $V(\Gamma_E(R_0))$ is the vertex set of $\Gamma_E(R_0)$. The result follows by counting the disjoint subsets.

Case II: When k is odd.

The vertex set of $\Gamma_E(R_0)$ is partitioned into the following subsets $V_1 = \{J^l\}$ with $1 \leq l \leq \frac{k-1}{2}$ and $V_i = \{J^i\}$ where $\frac{k-1}{2} < i \leq k-1$. Then clearly as in case I above, for each i, $V_1 \cap V_i = \emptyset$ and V_i are mutually disjoint. Moreover $V_1 \bigcup \cup_{i=\frac{k-1}{2}}^{k-1} \{V_i\} = V(\Gamma_E(R_0))$. The result follows by counting the disjoint subsets.

□

Corollary 3.2.10 *Let $R_0 = GR(p^{kr},\ p^k)$ where $k \geq 4$. Then the clique number,*

$$\omega(\Gamma_E(R_0)) = \begin{cases} \frac{k}{2}, & \textit{if } k \textit{ is even;} \\ \\ \frac{k+1}{2}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof.

It suffices to find a maximal complete subgraph of $\Gamma_E(R_0))$. Let s be a unit in R_0 and the elements of the vertex set of $\Gamma_E(R_0))$ be of the form p^l such that $\frac{k}{2} \leq l \leq k-1$ when k is even and $\frac{k-1}{2} \leq l \leq k-1$ when k is odd. We consider the following cases;

Case I: Let k be even.

We show that $\Gamma_E(R_0)$ has a maximal complete subgraph S with vertices $\{[p^l s] = [p^l]\}$ for $\frac{k}{2} \leq l \leq k-1$. Suppose on the contrary that S is not maximal in $\Gamma_E(R_0)$. Then there exists $S' \subset \Gamma_E(R_0)$ so that $S \subset S' \subseteq \Gamma_E((R_0)$. Without loss of generality, assume that $p^i \in V(S')$ where $\begin{cases} 0 < i < \frac{k}{2}, & \text{if } k \text{ is even ;} \\ 0 < i < \frac{k-1}{2}, & \text{if } k \text{ is odd.} \end{cases}$

So there exists some $j > i > 0$ so that $p^i.p^{k-1-j} = p^{k-1+i-j} \neq 0$ which implies that S' is not a complete subgraph, leading to a contradiction.

Case II: When k is odd:

By a similar argument as in Case I above, we can show that $\Gamma_E(R_0)$ contains a maximal complete subgraph with vertices $\{[p^l s] = [p^l]\}$, for $\frac{k-1}{2} \leq l \leq k-1$. □

Proposition 3.2.11 *Let $R_0 = GR(p^{kr},\ p^k),\ k \geq 4$. Then*

(i) diameter, $diam(\Gamma_E(R_0)) = 2$.

(ii) girth, $gr(\Gamma_E(R_0)) = 3$.

(iii) binding number, $b(\Gamma_E(R_0)) = \begin{cases} \frac{k-4}{k}, & \textit{if } k \textit{ is even;} \\ \frac{k-5}{k-1}, & \textit{if } k \textit{ is odd.} \end{cases}$

Proof.

Proofs to (i) and (ii) are easy to see.

To prove (iii), we consider the two separate cases when k is even and when k is odd respectively. By considering partitions in the proof of Corollary 3.2.9 if k is even, $|V_1| = \frac{k}{2}$ while $|N(V_1)| = \frac{k-4}{2}$ so that $b(\Gamma_E(R_0)) = \frac{k-4}{k}$. Finally, when k is odd, $|V_1| = \frac{k-1}{2}$ while $|N(V_1)| = \frac{k-5}{2}$ by partition in the proof of same Corollary 3.2.9. Then $b(\Gamma_E(R_0)) = \frac{|N(V_1)|}{|V_1|} = \frac{k-5}{k-1}$. □

Proposition 3.2.12 *Consider $R_0 = GR\left(p^{kr},\ p^k\right)$ for $k \geq 2$ and $r \geq 1$. Then*

$$\left|Aut\left(\Gamma\left(R_0\right)\right)\right| = \Pi_{l=2}^{k}\left(p^{(k-l)r}\left(p^r - 1\right)\right)!$$

Proof.

Let $\epsilon_1, \ldots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\bar{\epsilon_1}, \ldots, \bar{\epsilon_r} \in R_0/Z(R_0)$ forms a basis for $R_0/Z(R_0)$ regarded as a vector space over its prime subfield $GF(p)$. For each prime integer p, let $X = \{p, p^2, \ldots, p^{k-1}\}$ and $V_{\Sigma a_i \epsilon_i}$ where $a_i \in X$ be disjoint vertices, then routine enumeration yields

$$Aut\left(\Gamma\left(R_0\right)\right) = \Pi_{l=2}^{k} S_{p^{(k-l)r}(p^r-1)} \text{ for } 2 \leq l \leq k.$$

□

Chapter Summary

The remark and subsequent theorems, summarizes our characterization of the zero divisor graph, $\Gamma(R_0)$ of the Galois rings established in this chapter.

Remark 3.2.13

(a) The zero divisor graphs of a Galois ring $GR(p^{kr},\ p^k)$ is of infinite girth if the ring is one of the following:

i) $GR(p^r,\ p)$.

(ii) $\mathbb{Z}_4$.

(iii) $\mathbb{Z}_8$.

(iv) $\mathbb{Z}_9$.

(b) The zero divisor graph of a Galois ring is triangular if the ring is $GR\left(2^4,\ 2^2\right)$.

(c) The diameter of zero divisor graph of a Galois ring is zero if the ring is $\mathbb{Z}_4$.

(d) The diameter of a zero divisor graph of a Galois ring is infinite if the ring is $GR(p^r,\ p)$.

(e) There exists no Galois ring whose zero divisor graph is an $n-gon, n > 3$.

(f) The binding number $b\left(\Gamma(GR(p^{kr},\ p^k))\right)$ *is infinite if* $k = 1$.

Theorem 3.2.14 *Let $R_0 = GR\left(p^{kr},\ p^k\right)$. Then,*

$$\textit{(i)}\quad \Gamma(R_0) = \begin{cases} \emptyset, & \textit{if } R_0 = GR(p^r,\ p); \\ K_{p^r-1}, & \textit{if }\ R_0 = GR(p^{2r},\ p^2); \\ (p^{(\frac{k}{2})r} - 1) - \textit{partite}, & \textit{if }\ k \geq 3\ \textit{ is even}; \\ p^{(\frac{k-1}{2})r} - \ \textit{partite}, & \textit{if }\ k \geq 3\ \ \textit{is odd.} \end{cases}$$

$$\textit{(ii)}\quad \textit{diam}\,(\Gamma(R_0)) = \begin{cases} 0, & \textit{if } p = 2, k = 2, r = 1; \\ 1, & \textit{if } p = 3, k = 2, r = 1; \\ 2, & \textit{elsewhere}\ . \end{cases}$$

$$\textit{(iii)}\quad gr\,(\Gamma(R_0)) = \begin{cases} \infty, & \textit{if } p = 2, k = 2, 3, r = 1; \\ & \quad \textit{or }\ \textit{if } p = 3, k = 2, r = 1; \\ 3, & \textit{elsewhere.} \end{cases}$$

$$\textit{(iv)}\quad \omega(\Gamma(R_0)) = \begin{cases} (p^{(\frac{k}{2})r} - 1) - \ \textit{partite}, & \textit{if }\ k \geq 3\ \ \textit{is even}; \\ p^{(\frac{k-1}{2})r} - \ \textit{partite}, & \textit{if }\ k \geq 3\ \ \textit{is odd.} \end{cases}$$

$$\textit{(v)}\quad b\,(\Gamma(R_0)) = \begin{cases} \frac{p^{(\frac{k}{2})r}-2}{p^{(k-1)r}-p^{(\frac{k}{2})r}+1}, & \textit{if }\ k \geq 3\ \ \textit{is even}; \\ \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1)r}-p^{(\frac{k-1}{2})r}}, & \textit{if }\ k \geq 3\ \textit{is odd.} \end{cases}$$

Theorem 3.2.15 *Let $R_0 = GR\left(p^{kr},\ p^k\right)$. Then,*

$$\textit{(i)}\quad \Gamma_E(R_0) = \begin{cases} K_1, & \textit{if } charR_0 = p^2; \\ K_2, & \textit{if } charR_0 = p^3; \\ \frac{k}{2} - \ partite, & \textit{if } k \geq 4 \textit{ is even }; \\ \frac{k+1}{2} - \ partite, & \textit{if } k\ >\ 4 \textit{ is odd }. \end{cases}$$

$$\textit{(ii)}\quad diam\left(\Gamma_E(R_0)\right) = \begin{cases} 0, & \textit{if } charR_0 = p^2; \\ 1, & \textit{if } charR_0 = p^3; \\ 2, & elsewhere\ . \end{cases}$$

$$\textit{(iii)}\quad gr\left(\Gamma_E(R_0)\right) = \begin{cases} \infty, & \textit{if } \ charR_0 = p^i, i = 2, 3; \\ 3, & elsewhere. \end{cases}$$

$$\textit{(iv)}\quad b\left(\Gamma_E(R_0)\right) = \begin{cases} \frac{k-4}{k}, & \textit{if } k \geq 4 \ \textit{ is even,} \\ \frac{k-5}{k-1}, & \textit{if } k > 4 \ \textit{ is odd.} \end{cases}$$

$$\textit{(v)}\quad \omega(\Gamma_E(R_0)) = \begin{cases} \frac{k}{2}, & \textit{if } k \geq 4 \textit{ is even,} \\ \frac{k+1}{2}, & \textit{if } k > 4 \textit{ is odd.} \end{cases}$$

(vi) $Aut\left(\Gamma(R_0)\right) = S_{p^{(k-l)r}(p^r-1)},\ 2 \leq\ l \leq\ k.$

Chapter 4

A class of finite rings I

Throughout this chapter, R shall denote finite commutative rings with identity as constructed in Section 4.1 below. The structure of the groups of units of these rings are known (see [24]). In this chapter, we verify some of the algebraic properties of R and further, identify and investigate the invariant properties of the graphs of their zero divisors.

4.1 Construction I

Let R_0 be the Galois ring of the form $GR\left(p^{kr},\ p^k\right)$ where p is a prime integer and k and r are positive integers. For each $i = 1, \ldots, h$ let $u_i \in Z(R)$ and U be an h-dimensional R_0-module generated by $\{u_1, \ldots, u_h\}$. Then $R = R_0 \oplus U$ is an additive group. On this group, define multiplication by the following relation;

(i) If $k = 1, 2$, then $pu_i = u_i u_j = u_j u_i = 0,\ u_i r_0 = (r_0)^{\sigma_i} u_i$ and

(ii) If $k \geq 3$, then $p^{k-1}u_i = 0,\ u_i u_j = p^2\gamma_{ij},\ u_i^k = u_i^{k-1}u_j = u_i u_j^{k-1} = 0$;

$u_i r_0 = (r_0)^{\sigma_i} u_i$, where $r_0, \gamma_{ij} \in R_0$, $1 \leq i, j \leq h$ and σ_i is the automorphism associated with u_i. Further, let the generators $\{u_i\}$ for U satisfy the additional condition that if $u_i \in U$, then $pu_i = u_i u_j = 0$.

By this multiplication on R, it is easy to see that if $r_0 + \Sigma_{i=1}^h \lambda_i u_i$ and $s_0 + \Sigma_{i=1}^h \gamma_i u_i$, where r_0, $s_0 \in R_0, \gamma_i, \lambda_i \in F_0 \cong R_0/pR_0$ are elements of R, then $\left(r_0 + \Sigma_{i=1}^h \lambda_i u_i\right)\left(s_0 + \Sigma_{i=1}^h \gamma_i u_i\right) = r_0 s_0 + \Sigma_{i=1}^h \left((r_0 + pR_0)\gamma_i + \lambda_i (s_0 + pR_0)^{\sigma_i}\right) u_i$.

We verify that this multiplication makes R a ring with the identity element equal to $(1, 0, \ldots, 0)$.

Let $r_0 + \Sigma_{i=1}^h \lambda_i u_i \in R$ with $r_0 \in R_0$ and $\lambda_i \in F_0 \cong R_0/pR_0$, then we need to find $s_0 + \Sigma_{i=1}^h \gamma_i u_i$ with $s_0 \in R_0$ and $\gamma_i \in F_0$ such that $\left(r_0 + \Sigma_{i=1}^h \lambda_i u_i\right)\left(s_0 + \Sigma_{i=1}^h \gamma_i u_i\right) = \left(s_0 + \Sigma_{i=1}^h \gamma_i u_i\right)\left(r_0 + \Sigma_{i=1}^h \lambda_i u_i\right) = r_0 + \Sigma_{i=1}^h \lambda_i u_i$.

Now if $r_0 s_0 + \Sigma_{i=1}^h \left((r_0 + pR_0)\gamma_i + \lambda_i (s_0 + pR_0)^{\sigma_i}\right) u_i = r_0 + \Sigma_{i=1}^h \lambda_i u_i$, then $r_0 s_0 = r_0$ and $\Sigma_{i=1}^h \left((r_0 + pR_0)\gamma_i + \lambda_i (s_0 + pR_0)^{\sigma_i}\right) u_i = \Sigma_{i=1}^h \lambda_i u_i$. So $((r_0 + pR_0)\gamma_i)u_i = 0_R$, and $s_0 = 1_{R_0}$ for each $i = 1, \ldots, h$. Since $u_i \neq 0, (r_0 + pR_0)\gamma_i = 0_{F_0}$. But $r_0 \in R_0$, so $\gamma_i = 0_{F_0}$ for each $i = 1, \ldots, h$. Thus $s_0 + \Sigma_{i=1}^h \gamma_i u_i = (1, 0, \ldots, 0)$. Similarly, we can show that $(s_0 + \Sigma_{i=1}^h \gamma_i u_i)(r_0 + \Sigma_{i=1}^h \lambda_i u_i) = r_0 + \Sigma_{i=1}^h \lambda_i u_i$ implies that $s_0 + \Sigma_{i=1}^h \gamma_i u_i = (1, 0, \ldots, 0)$.

Now, we prove that multiplication is associative. Suppose r_0, s_0, $t_0 \in R_0$ and λ_i, γ_i, $\kappa_i \in F_0$, let $r_0 + \Sigma_{i=1}^h \lambda_i u_i$, $s_0 + \Sigma_{i=1}^h \gamma_i u_i$, $t_0 + \Sigma_{i=1}^h \kappa_i u_i \in R$. Then

$(r_0 + \Sigma_{i=1}^h \lambda_i u_i)((s_0 + \Sigma_{i=1}^h \gamma_i u_i)(t_0 + \Sigma_{i=1}^h \kappa_i u_i))$

$= (r_0 + \Sigma_{i=1}^h \lambda_i u_i)(s_0 t_0 + \Sigma_{i=1}^h ((s_0 + pR_0)\kappa_i + \gamma_i (t_0 + pR_0)^{\sigma_i}) u_i)$

$= r_0 s_0 t_0 + \Sigma_{i=1}^h ((r_0 + pR_0)((s_0 + pR_0)\kappa_i + \gamma_i (t_0 + pR_0)^{\sigma_i}) + \lambda_i (s_0 t_0 + pR_0)^{\sigma_i}) u_i$

$= r_0 s_0 t_0 + \Sigma_{i=1}^h ((r_0 s_0 + pR_0)\kappa_i + ((r_0 + pR_0)\gamma_i + \lambda_i (s_0 + pR_0)^{\sigma_i})(t_0 + pR_0)^{\sigma_i}) u_i$

$= (r_0s_0 + \Sigma_{i=1}^{h}((r_0 + pR_0)\gamma_i + \lambda_i(s_0 + pR_0)^{\sigma})u_i)(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i)$

$= ((r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)(s_0 + \Sigma_{i=1}^{h}\gamma_iu_i))(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i).$

Moreover,

$(r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)((s_0 + \Sigma_{i=1}^{h}\gamma_iu_i) + (t_0 + \Sigma_{i=1}^{h}\kappa_iu_i))$

$= (r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)(s_0 + t_0 + \Sigma_{i=1}^{h}(\gamma_i + \kappa_i)u_i)$

$= r_0(s_0 + t_0) + \Sigma_{i=1}^{h}((r_0 + pR_0)(\gamma_i + \kappa_i) + \lambda_i((s_0 + t_0 + pR_0)^{\sigma_i})u_i$

$= r_0s_0 + \Sigma_{i=1}^{h}((r_0 + pR_0)\gamma_i + \lambda_i(s_0 + pR_0)^{\sigma_i}) + r_0t_0 + \Sigma_{i=1}^{h}((r_0 + pR_0)\kappa_i + \lambda_i(t_0 + pR_0)^{\sigma_i})$

$= (r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)(s_0 + \Sigma_{i=1}^{h}\gamma_iu_i) + (r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i)$, which shows that the left distributive law holds in R. Similarly, we can show that,

$((r_0 + \Sigma_{i=1}^{h}\lambda_iu_i) + (s_0 + \Sigma_{i=1}^{h}\gamma_iu_i))(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i)$

$= (r_0 + \Sigma_{i=1}^{h}\lambda_iu_i)(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i) + (s_0 + \Sigma_{i=1}^{h}\gamma_iu_i)(t_0 + \Sigma_{i=1}^{h}\kappa_iu_i)$, so that the right distributive law also holds. Clearly R is a ring with identity.

We now discuss some properties of R.

Lemma 4.1.1 *R is commutative if and only if $\sigma_i = id_{R_0}$, the identity automorphism, for all $i = 1, \ldots, h$.*

Proof.

If $\sigma_i = id_{R_0}$, then commutativity of R follows from the definition of multiplication. Conversely, suppose R is commutative. Then for each $a_0, b_0 \in R_0, \alpha_i, \beta_i \in R_0/pR_0$, we have that $\left(a_0 + \Sigma_{i=1}^{h}\alpha_iu_i\right)\left(b_0 + \Sigma_{i=1}^{h}\beta_iu_i\right) = \left(b_0 + \Sigma_{i=1}^{h}\beta_iu_i\right)\left(a_0 + \Sigma_{i=1}^{h}\alpha_iu_i\right)$. This implies that,

$$a_0b_0 + \Sigma_{i=1}^{h}\left[\left(a_0 + pR_0\right)\beta_i + \alpha_i\left(b_0 + pR_0\right)^{\sigma_i}\right]u_i = b_0a_0 + \Sigma_{i=1}^{h}\left[\left(b_0 + pR_0\right)\alpha_i + \beta_i\left(a_0 + pR_0\right)^{\sigma_i}\right]u_i.$$

Commutativity of R then implies that $a_0b_0 = b_0a_0$ which requires that,

$$\Sigma_{i=1}^{h}\left[\left(a_0 + pR_0\right)\beta_i + \alpha_i\left(b_0 + pR_0\right)^{\sigma_i}\right]u_i = \Sigma_{i=1}^{h}\left[\left(b_0 + pR_0\right)\alpha_i + \beta_i\left(a_0 + pR_0\right)^{\sigma_i}\right]u_i.$$

Which further implies that $\alpha_i\,(b_0 + pR_0)^{\sigma_i} - (b_0 + pR_0)\,\alpha_i = \beta_i\,(a_0 + pR_0)^{\sigma_i} - (a_0 + pR_0)\,\beta_i$. Since $\alpha_i \neq \beta_i$, then for the equality to hold then α_i must be the identity in R_0, i. e. $\sigma_i = id_{R_0}$, for all $i = 1, \ldots, h$. □

Proposition 4.1.2 *If $k = 1$ or 2, then R is a ring in which the multiplication of any two zero divisors is zero, that is $(Z(R))^2 = (0)$.*

Proof.

Follows from the definition of multiplication on R. □

Remark 4.1.3 *Such rings with property of Proposition 4.1.2 are well known to be completely primary finite rings (See Alkhamees [3]).*

4.2 Rings of characteristic p

Let $R_0 = GF(p^r)$ and $F = R_0/pR_0$ so that $U = F^h$ is an R_0-module generated by $u_1, u_2, \ldots, u_h$. On the additive group $R = R_0 \oplus U = R_0 \oplus F^h$, define multiplication on R as follows:

$(r_0, r_1, \ldots, r_h)(s_0, s_1, \ldots, s_h) = (r_0 s_0, r_0 s_1 + r_1 s_0, \ldots, r_0 s_h + r_h s_0)$.

This multiplication turns R into a ring with identity $(1, 0, \ldots, 0)$.

Proposition 4.2.1 *If in the Construction I, R is a ring of characteristic p, then*

(i) $|\Gamma(R)| = p^{rh} - 1$.

(ii) $\Gamma(R)$ *is complete.*

(iii) $\Gamma(R) = K_{p^{rh}-1}$.

(iv) $diam(\Gamma(R)) = 1$.

(v) $gr(\Gamma(R)) = \begin{cases} \infty, & \textit{if } r = 1,\ h = 1 \textit{ and } p = 2,\ 3; \\ 3, & \textit{elsewhere.} \end{cases}$

(vi) The binding number $b(\Gamma(R)) = \infty$.

Proof.

(i) By the above construction, we have that $R_0 = GF(p^r)$ and $F = R_0/pR_0$. Let $U = F^h$ be an R_0-module generated by $u_1, \dots, u_h$ so that $R = R_0 \oplus U$ is an additive group. It is clear that $Z(R) = R_0u_1 \oplus R_0u_2 \oplus \cdots \oplus R_0u_h$ and every non zero element in $Z(R)$ is of the form $(0, r_1, r_2, \dots, r_h)$. We show that any element not contained in $Z(R)$ is invertible. So let $(r_0, r_1, r_2, \dots, r_h) \notin Z(R)$. Choose an element say $(s_0, s_1, s_2, \dots, s_h) \notin Z(R)$ such that $(r_0, r_1, r_2, \dots, r_h)(s_0, s_1, s_2, \dots, s_h) = (1, 0, 0, \dots, 0)$. This implies that $r_0s_0 = 1$, thus $s_0 = r_0^{-1}$ and $r_0s_i + r_is_0 = 0$, which implies that $s_i = -r_ir_0^{-2}$ for $1 < i \leq h$. Since this holds in the reverse order, we have established that

$$(r_0, r_1, r_2, \dots, r_h)^{-1} = (r_0^{-1}, -r_1r_0^{-2}, -r_2r_0^{-2}, \dots, -r_hr_0^{-2}).$$

Since $|R| = |R_0||U| = p^r.p^{hr} = p^{(h+1)r}$ so that $|Z(R)| = p^{rh}$ and $V(\Gamma(R)) = Z(R) - \{0\}$, then $|\Gamma(R)| = p^{rh} - 1$ which establishes (i).

(ii) To establish this, note that the product of every pair $(0, r_1, r_2, \dots, r_h)$, $(0, s_1, s_2, \dots, s_h) \in Z(R) - \{0\}$ is equal to zero so that every pair of vertices in $V(\Gamma(R))$ are adjacent. Hence $\Gamma(R)$ is complete.

(iii) This follows from (i) and (ii).

(iv) Follows from (ii) and (iii).

Alternatively, note that $diam(\Gamma(R)) = Sup\{d(x,y)|x,y \in \Gamma(R)\}$. Since $V(\Gamma(R)) = Z(R)-\{0\}$ and for all distinct $x,y \in Z(R)-\{0\}, xy = 0,\ \Gamma(R)$ is complete (see Definition 1.9.12), hence $d(x,y) = 1$. So $Sup\{d(x,y)\} = 1$ for all $x,y \in \Gamma(R)$ which implies that $diam(\Gamma(R)) = 1$.

(v) This follows from (iii).

Alternatively we note that $\Gamma(R) = K_{p^{rh}-1}$ is complete. Then when $r = 1,\ h = 1$ and $p = 2$ or 3 then $(p^{rh} - 1) \leq 2$. So $gr(\Gamma(R)) = \infty$. Otherwise, for all $(p^{rh} - 1) > 2$ we have by Diestel [18] that $gr(\Gamma(R)) = 2\ diam(\Gamma(R)) + 1$. Since $diam(\Gamma(R)) = 1$, the result readily follows.

(vi) Since the set S of minimal degree in $\Gamma(R))$ is empty, $b(\Gamma(R)) = \infty$.

□

4.3 Rings of characteristic p^2

Let $R_0 = GR(p^{2r},\ p^2)$ and $F = R_0/pR_0$ so that $U = F^h$ is an R_0-module generated by $u_1, u_2, \ldots, u_h$ on the additive group $R = R_0 \oplus U = R_0 \oplus F^h$. Define multiplication as follows: $(r_0, \overline{r}_1, \ldots, \overline{r}_h)(s_0, \overline{s}_1, \ldots, \overline{s}_h) = (r_0 s_0, r_0\overline{s}_1 + \overline{r}_1 s_0, \ldots r_0\overline{s}_h + \overline{r}_h s_0)$. This multiplication turns R into a ring with identity $(1, \overline{0}, \ldots, \overline{0})$.

Proposition 4.3.1 *Let R be the ring of Construction I whose $charR = p^2$. Then the following hold:*

(i) $|\Gamma(R)| = p^{(h+1)r} - 1$.

(ii) $\Gamma(R)$ *is complete.*

(iii) $\Gamma(R) = K_{p^{(h+1)r}-1}$.

(iv) $diam((\Gamma(R))) = 1$.

(v) $gr(\Gamma(R)) = \begin{cases} \infty, & \text{if } r = 1,\ h = 0 \text{ or } 1 \text{ and } p = 2 \text{ or } 3; \\ 3, & \text{elsewhere.} \end{cases}$

(vi) The binding number $b(\Gamma(R)) = \infty$.

Proof.

(i) By the construction, $x \in Z(R) - \{0\}$ if and only if x is of the form $(0, \overline{r_1}, \overline{r_2}, \ldots, \overline{r_h})$. Now let $x = (r_0, \overline{r_1}, \overline{r_2}, \ldots, \overline{r_h}) \notin Z(R)$ be an element in R, then x is invertible and indeed x^{-1} is $(s_0, \overline{s_1}, \ldots, \overline{s_h})$ such that $s_0 = r_0^{-1}$ and $\overline{s_i} = -\overline{r_i} r_0^{-2}$ for $1 \leq i \leq h$. Since $|R| = |R_0||U| = p^{2r}.p^{hr} = p^{(h+2)r}$. Thus $|Z(R)| = p^{(h+1)r}$ and $\Gamma(R) = Z(R) - \{0\}$, then $|\Gamma(R)| = p^{(h+1)r} - 1$.

(ii) For all $x, y \in Z(R) - \{0\}, xy = 0$. Then as in proof of part (ii) of Proposition 4.2.1, $\Gamma(R)$ is complete.

(iii) This also follows from (i) and (ii), i.e $\Gamma(R) = K_{p^{(h+1)r}-1}$.

(iv) It is clear that for all $x, y \in \Gamma(R)$, $d(x, y) = 1$. So $Sup\{d(x, y)\} = 1$ for all $x, y \in \Gamma(R)$. Therefore the result is immediate.

(v) When $r = 1$, $h = 0$ and $p = 2$ or 3 then $\Gamma(R) = K_n$ is complete. Since $n = (p^{(h+1)r} - 1) \leq 2$, $\Gamma(R)$ has no circles. So $gr(\Gamma(R)) = \infty$. Otherwise, for all r, $h \geq 1$, $n = (p^{(h+1)r} - 1) > 2$. So the completeness of $\Gamma(R)$ implies that $gr(\Gamma(R)) = 2diam(\Gamma(R)) + 1 = 3$, since $diam(\Gamma(R)) = 1$.

(vi) Similar to Proposition 4.2.1.

□

Corollary 4.3.2 *Let R be a ring in Construction I satisfying $(Z(R))^2 = (0)$. The graph $\Gamma(R)$ is triangular if R is either of the following.*

(i) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(ii) $\mathbb{F}_4 \oplus \mathbb{F}_4$.

(iii) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Proof.

A zero divisor graph is triangular if $\Gamma(R) = K_3$. Let R be a ring described by Construction I with $char(R) = p$ or p^2, then $\Gamma(R) = K_{p^{hr}-1}$ or $\Gamma(R) = K_{p^{(h+1)r}-1}$ respectively. It suffices to find values of p, r and h for which $p^{hr} - 1$ or $p^{(h+1)r} - 1$ equals 3. Now let R be of characteristic $p = 2$. Then (i) holds for $r = 1$ and $h = 2$. (ii) holds when $r = 2,\ h = 1$. When R is of characteristic p^2, then $\Gamma(R)$ is clearly triangular in case (iii), when $r = 1,\ h = 1$. □

4.4 Rings of characteristic $p^k,\ k \geq 3$

Lemma 4.4.1 *Let $R_0 = GR(p^k,\ p^k),\ k \geq 3$ and $R = R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$. Then, $Z(R) = pR_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$ and $ann(Z(R)) = p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$ for $n \geq 2$.*

Proof.

Let $x \in ann(Z(R))$, then $x \in R$, so that $Z(R)x = xZ(R) = (0)$. So let

$x = (r_0, \bar{r}_1, \ldots, \bar{r}_h) \in Z(R)$ and $u = (ps_0, \bar{s}_1, \ldots, \bar{s}_h) \in Z(R)$. Now

$xu = (r_0, \bar{r}_1, \ldots, \bar{r}_h)(ps_0, \bar{s}_1, \ldots, \bar{s}_h) = (r_0 ps_0, r_0\bar{s}_1 + \bar{r}_1 ps_0, \ldots, r_0\bar{s}_h + \bar{r}_h ps_0)$

$= (ps_0 r_0, ps_0\bar{r}_1 + \bar{s}_1 r_0, \ldots, ps_0\bar{r}_h + \bar{s}_h r_0) = ux = (0, \bar{0}, \ldots, \bar{0})$.

Thus $r_0 ps_0 = ps_0 r_0 = 0$, implies that $s_0 r_0 \in p^{n-1}R_0$.

Since $s_0 \in R_0$, then $r_0 \in p^{n-1}R_0$. Moreover, $r_0\bar{s}_1 + \bar{r}_1 ps_0 = \cdots = r_0\bar{s}_h + \bar{r}_h ps_0 = \bar{0}$. For each $i = 1, \ldots, h$, $r_0\bar{s}_i + \bar{r}_i ps_0 = \bar{0}$. This implies that, $r_0\bar{s}_i + \bar{r}_i ps_0 \in pR_0$. Since $r_0 \in p^{n-1}R_0$, $\bar{s}_i \in R_0/pR_0$, then $r_0\bar{s}_i \in pR_0$ and $\bar{r}_i ps_0 \in pR_0$ so that $\bar{r}_i \in R_0/pR_0$. This shows that $x = (r_0, \bar{r}_1, \ldots, \bar{r}_h) \in p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$ which implies that $ann(Z(R)) \subseteq p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$.

Conversely, let $x \in p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$. Notice that $Z(R)^{n-1} = p^{n-1}R_0$ so that $(p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0)Z(R) = Z(R)(p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0) = (Z(R))^n = (0)$, hence $p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0 \subseteq ann(Z(R))$.

□

Lemma 4.4.2 *Let* $R_0 = GR(p^k,\ p^k)$, $k \geq 3$ *and let* $R = R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$. *Then,*

$Z(R) = pR_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$, $ann(Z(R)) = p^{n-1}R_0 \oplus R_0/pR_0 \oplus \cdots \oplus R_0/pR_0$ *and* $(Z(R))^{n-1} = p^{n-1}R_0$. *Moreover, when*

(i) $x \in ann(Z(R))$, *then* $deg(x) = \mid Z(R) \mid - 2$.

(ii) $y \in Z(R)$ *but* $y \notin ann(Z(R))$, *then* $deg(y) = \mid ann(Z(R)) \mid - 1$.

PROOF.

(i) Clearly $|ann(Z(R))| = p^{h+1}$. Now since x is adjacent to all the other vertices except 0 and itself, we have $deg(x) = |Z(R)| - 2$.

(ii) Let $y \in Z(R)$ but $y \notin ann(Z(R))$. Suppose $z \notin ann(Z(R))$, we claim that z is not adjacent to y. Suppose it is, then $zy = 0$. Now z is of the form $(1 + p^{n-1}r_0, \bar{r}_1, \ldots, \bar{r}_h)$ and y is of the form $(ps_0, \bar{s}_1, \ldots, \bar{s}_h)$ so that $(1 + p^{n-1}r_0, \bar{r}_1, \ldots, \bar{r}_h)(ps_0, \bar{s}_1, \ldots, \bar{s}_h) = (ps_0, \bar{s}_1, \ldots, \bar{s}_h) \neq (0, \bar{0}, \ldots, \bar{0})$, a contradiction and we are done.

□

Proposition 4.4.3 *Let R be a ring in Construction I. If $k \geq 3$, then R is in the class of completely primary finite rings of characteristic p^k satisfying;*

(i) $Z(R) = pR_0 \oplus U$.

(ii) $(Z(R))^{k-1} = p^{k-1}R_0$.

(iii) $(Z(R))^k = (0)$.

PROOF.

It is well known that R_0 is a coefficient subring of R with same identity element and of same characteristic. To show that $Z(R) = pR_0 \oplus U$, we prove that all elements which lie outside $Z(R)$ are invertible. Consider $b \in R_0$ such that $b \notin pR_0$ and $t \in Z(R)$. Then, $(b+t)^{p^r} = b^{p^r} + t_1$, where $t_1 \in Z(R)$. But $b^{p^r} + t_1 = b + t_2$, where $t_2 \in Z(R)$. Now, $(b+t_2)^{p^r-1} = 1 + t_3$, where $t_3 \in Z(R)$, and $(1+t_3)^{p^{k-1}} = 1$. So $\left(\left((b+t)^{p^r}\right)^{p^r-1}\right)^{p^{k-1}} = 1$, which shows that $b+t$ has an inverse. Further,

$|Z(R)| = p^{(h+k-1)r}$ and $|\left(R_0/pR_0\right)^\star + Z(R)| = (p^r - 1)\left(p^{(h+k-1)r}\right)$, so that $(R_0/pR_0)^\star + Z(R) = R - Z(R)$, which shows that all the elements which lie outside $Z(R)$ are invertible. The multiplication on R yields that $\left(Z(R)\right)^{k-1} = p^{k-1}R_0$ and $Z(R)\left(p^{k-1}R_0\right) = \left(p^{k-1}R_0\right)Z(R) = (0)$. Thus $\left(Z(R)\right)^k = (0)$.

Since $RZ(R) = Z(R) \subseteq Z(R)$, the set $Z(R)$ is an ideal. Its uniqueness and maximality follows from the fact that any other ideal distinct from $Z(R)$ contains a unit and is therefore the whole ring R. □

Proposition 4.4.4 *Let R_0 be a Galois ring of the form $GR(p^k,\ p^k)$, $k \geq 3$ and $U = \Sigma_{i=1}^h \oplus (R_0/pR_0)^i$ considered as an $h-$dimensional R_0-module. On the additive group $R = R_0 \oplus U$, define multiplication by*

$(r_0, r_1, \ldots, r_h)(s_0, s_1, \ldots, s_h) = (r_0s_0, r_0s_1 + r_1s_0, \ldots, r_0s_h + r_hs_0)$.

$$\textit{Then } \Gamma(R) = \begin{cases} p^{\frac{k}{2}+h} \ - \ \textit{partite}, & k \textit{ is even };\\ \\ p^{\frac{k-1}{2}+h} \ - \ \textit{partite}, & k \textit{ is odd.}\end{cases}$$

Proof.

Case I: k is an even integer. Partition $Z(R)^\star$ into the following subsets.

$V_{(1,r_1,\ldots,r_h)} = Z(R)^\star \backslash \left\{(j(p^{\frac{k}{2}}), r_1, \ldots, r_h) \text{ where } 1 \leq j \leq p^{\frac{k}{2}} - 1\right\}$.

$V_{(j,r_1,\ldots,r_h)} = \left\{(j(p^{\frac{k}{2}}), r_1, \ldots, r_h)\right\}$ where $1 \leq j \leq p^{\frac{k}{2}} - 1$.

$V_{(0,r_1,\ldots,r_h)} = \{(0, r_1, \ldots, r_h)\}$ for at least one $r_i \neq 0$ and for $1 \leq i \leq h$.

Clearly all the above defined sets are nonempty and each set contains nonadjacent vertices. Thus, $V_{(1,r_1,\ldots,r_h)} \cap V_{(j,r_1,\ldots,r_h)} = \emptyset$; $V_{(1,r_1,\ldots,r_h)} \cap V_{(0,r_1,\ldots,r_h)} = \emptyset$ and $V_{(j,r_1,\ldots,r_h)} \cap V_{(l,r_1,\ldots,r_h)} = \emptyset$ for all $j \neq l,\ 1 \leq j,\ l \leq p^{\frac{k}{2}} - 1$.

Finally observe that $Z(R)^\star = V_{(1,r_1,\ldots,r_h)} \cup \left(\cup_{j=1}^{p^{\frac{k}{2}-1}}\{V_{(j,r_1,\ldots,r_h)}\}\right) \cup V_{(0,r_1,\ldots,r_h)}$. Now, $\left|V_{(1,r_1,\ldots,r_h)}\right| = 1$; $\left|\cup_{j=1}^{p^{\frac{k}{2}-1}}\{V_{(j,r_1,\ldots,r_h)}\}\right| = \left(p^{\frac{k}{2}} - 1\right)p^h$ and $|V_{(0,r_1,\ldots,r_h)}| = p^h - 1$. Then,

$$\left| V_{(1,r_1,\ldots,r_h)} \cup \left(\cup_{j=1}^{p^{\frac{k}{2}-1}} \{ V_{(j,r_1,\ldots,r_h)} \} \right) \cup V_{(0,r_1,\ldots,r_h)} \right| = 1 + \left(p^{\frac{k}{2}} - 1\right) p^h + p^h - 1 = p^{\frac{k}{2}+h}.$$

Thus $\Gamma(R)$ is $p^{\frac{k}{2}+h} -$ partite if k is even.

Case II: k is an odd integer.

Partition $Z(R)^\star$ into the following subsets.

$V_{(1,r_1,\ldots,r_h)} = Z(R)^\star \backslash \left\{ ((j-1)(p^{\frac{k+1}{2}}), r_1, \ldots, r_h) \text{ where } 2 \le j \le p^{\frac{k-1}{2}} \right\}.$

$V_{(j,r_1,\ldots,r_h)} = \left\{ \left((j-1)(p^{\frac{k+1}{2}}), r_1, \ldots, r_h \right) \right\}$ where $2 \le j \le p^{\frac{k-1}{2}}$.

$V_{(0,r_1,\ldots,r_h)} = \{(0, r_1, \ldots, r_h)\}$ for at least one $r_i \neq 0$ and for $1 \le i \le h$.

These sets are nonempty and each set contains nonadjacent vertices.

That is, $V_{(1,r_1,\ldots,r_h)} \cap V_{(j,r_1,\ldots,r_h)} = \emptyset$; $V_{(1,r_1,\ldots,r_h)} \cap V_{(0,r_1,\ldots,r_h)} = \emptyset$, and

$V_{(j,r_1,\ldots,r_h)} \cap V_{(l,r_1,\ldots,r_h)} = \emptyset$ for all $j \neq l$, $2 \le j, l \le p^{\frac{k-1}{2}}$. Moreover,

$Z(R)^\star = V_{(1,r_1,\ldots,r_h)} \cup \left(\cup_{j=2}^{p^{\frac{k-1}{2}}} \{ V_{(j,r_1,\ldots,r_h)} \} \right) \cup V_{(0,r_1,\ldots,r_h)}$. Now, $\left| V_{(1,r_1,\ldots,r_h)} \right| = 1$;

$\left| \cup_{j=2}^{p^{\frac{k-1}{2}}} \{ V_{(j,r_1,\ldots,r_h)} \} \right| = p^h \left(p^{\frac{k-1}{2}} - 1 \right)$ and $\left| V_{(0,r_1,\ldots,r_h)} \right| = p^h - 1$. Thus,

$$\left| V_{(1,r_1,\ldots,r_h)} \cup \left(\cup_{j=2}^{p^{\frac{k-1}{2}}} \{ V_{(j,r_1,\ldots,r_h)} \} \right) \cup V_{(0,r_1,\ldots,r_h)} \right| = 1 + p^h \left(p^{\frac{k-1}{2}} - 1 \right) + p^h - 1 = p^{\frac{k-1+2h}{2}}.$$

Therefore $\Gamma(R)$ is $p^{\frac{k-1+2h}{2}} -$ partite if k is odd. □

Example 4.4.5 *Let R_0 be a Galois ring of the form $GR(p^k,\ p^k)$, $k \ge 3$ and $U = \Sigma_{i=1}^{h} \oplus (R_0/pR_0)^i$ considered as an $h-$dimensional R_0-module*

On the additive group $R = R_0 \oplus U$, define multiplication by

$(r_0, r_1, \ldots, r_h)(s_0, s_1, \ldots, s_h) = (r_0 s_0, r_0 s_1 + r_1 s_0, \ldots, r_0 s_h + r_h s_0).$

$$\textit{Then } \Gamma(R) = \begin{cases} p^{\frac{k}{2}+h} - \textit{ partite}, & k \textit{ is even };\\ p^{\frac{k-1}{2}+h} - \textit{ partite}, & k \textit{ is odd}. \end{cases}$$

Choose $R_0 = \mathbb{Z}_{16}$, $h = 1$, $p = 2$, $k = 4$. Then $R = \mathbb{Z}_{16} \oplus \mathbb{Z}_{16}/2\mathbb{Z}_{16}$ and $Z(R)^\star =$

$\{(0,\bar{1}), (2,\bar{0}), (2,\bar{1}), (4,\bar{0}), (4,\bar{1}), (6,\bar{0}), (6,\bar{1}), (8,\bar{0}), (8,\bar{1}), (10,\bar{0}), (10,\bar{1}), (12,\bar{0}), (12,\bar{1}), (14,\bar{0}), (14,\bar{1})\}$

Now $V_{(0,1)} = \{(0, \bar{1})\}$, $V_{(1,r_1)} = \{(2, \bar{0}), (2, \bar{1}), (6, \bar{0}), (6, \bar{1}), (10, \bar{0}), (10, \bar{1}), (14, \bar{0}), (14, \bar{1})\}$, $V_{(1,0)} = \{(4, \bar{0})\}$, $V_{(1,1)} = \{(4, \bar{1})\}$, $V_{(2,0)} = \{(8, \bar{0})\}$, $V_{(2,1)} = \{(8, \bar{1})\}$, $V_{(3,0)} = \{(12, \bar{0})\}$ *and* $V_{(3,1)} = \{(12, \bar{1})\}$.

Clearly, each pair of sets is disjoint and each set contains nonadjacent vertices.

Then $V_{(1,r_1)} \cup V_{(1,0)} \cup V_{(1,1)} \cup V_{(2,0)} \cup V_{(2,1)} \cup V_{(3,0)} \cup V_{(3,1)} \cup V_{(0,1)} = Z(R)^\star$. *So* $\Gamma(R)$ *is* $8 \, - \,$ *partite (See Figure 4.1), with* $diam(\Gamma(R)) = 2$, $gr(\Gamma(R)) = 3$ *and* $b(\Gamma(R)) = \frac{7}{8}$

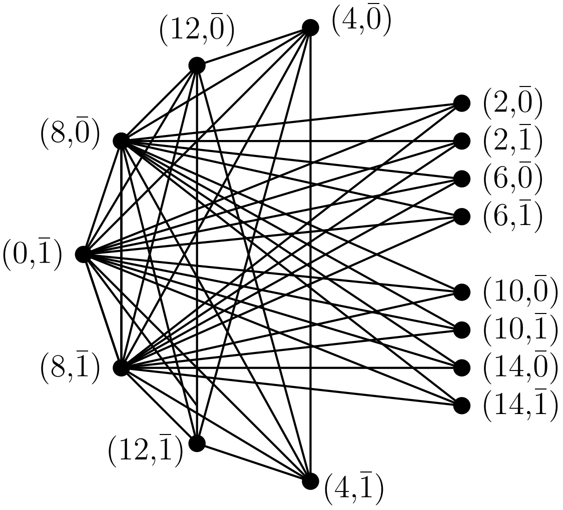


Figure 4.1: The zero divisor graph of $\mathbb{Z}_{16} \oplus \mathbb{Z}_{16}/2\mathbb{Z}_{16}$

Example 4.4.6 *Choose* $R_0 = \mathbb{Z}_8$; $h = 1$, $p = 2$, $k = 3$. *Then,* $R = \mathbb{Z}_8 \oplus \mathbb{Z}_8/2\mathbb{Z}_8$ *and* $Z(R)^\star = \{(2, \bar{0}), (2, \bar{1}), (4, \bar{0}), (4, \bar{1}), (6, \bar{0}), (6, \bar{1}), (0, \bar{1})\}$. *Now* $V_{(1,r_1)} = \{(2, \bar{0}), (2, \bar{1}), (6, \bar{0}), (6, \bar{1})\}$, $V_{(2,0)} = \{(4, \bar{0})\}$, $V_{(2,1)} = \{(4, \bar{1})\}$ *and* $V_{(0,1)} = \{(0, \bar{1})\}$.

Notice that each pair of sets is disjoint , and each set contains nonadjacent vertices.

$V_{(1,r_1)} \cup V_{(2,0)} \cup V_{(2,1)} \cup V_{(0,1)} = Z(R)^\star$.

Thus $\Gamma(R)$ *is* $4 \, - \,$ *partite (See Figure 4.2), with* $diam(\Gamma(R)) = 2$,

$gr(\Gamma(R)) = 3$ *and* $b(\Gamma(R)) = \frac{3}{4}$.

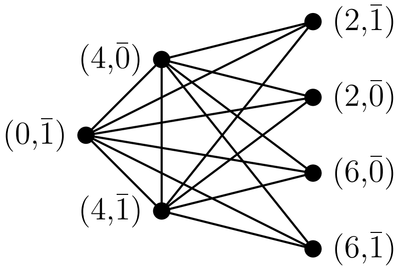


Figure 4.2: The zero divisor graph of $\mathbb{Z}_8 \oplus \mathbb{Z}_8/2\mathbb{Z}_8$

Proposition 4.4.7 *Let R_0 be the Galois ring of the form $GR(p^{kr},\ p^k)$, where p is a prime integer and k and r are positive integers. For each $i = 1, \dots h$, let $u_i \in Z(R)$ such that U is an h-dimensional R_0-module generated by $\{u_i, \dots, u_h\}$ so that $R = R_0 \oplus U$ is an additive group. On this group, define multiplication by the following relations;*

(i) If $k = 1, 2$, then $pu_i = u_iu_j = u_ju_i = 0$; $u_ir_0 = r_0u_i$.

(ii) If $k \geq 3$, then $p^{k-1}u_i = 0, u_iu_j = p^2\gamma_{ij},\ u_i^k = u_i^{k-1}u_j = u_iu_j^{k-1} = 0, u_ir_0 = r_0u_i$ where $r_0, \gamma_{ij} \in R_0,\ 1 \leq i, j \leq h$. In addition, if u is restricted to U then the order of u is p.

$$\text{Then } \ \Gamma(R) = \begin{cases} p^{(\frac{k}{2}+h)r} \ - \ \textit{partite}, & \textit{if } k \textit{ is even;} \\ p^{(\frac{k-1+2h}{2})r} \ - \ \textit{partite}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof.

Let $\lambda_1, \dots, \lambda_r \in R_0$ with $\lambda_1 = 1$ such that $\bar{\lambda_1}, \dots, \bar{\lambda_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield F_p. Since the two cases do not overlap, we treat them in turn.

CaseI: k is an even integer:

Let $X_{i,s} = \{\Sigma_{i=1}^{r} a_i\lambda_i + \Sigma_{s=1}^{h}\lambda_s u_s\}$ where $a_i \in \{0, j(p^{\frac{k}{2}})\}$ and $1 \leq j \leq p^{\frac{k}{2}} - 1$. Then $Z(R)^{\star}$ is partitioned into the following mutually disjoint subsets

$V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s} = X_{i,s}\backslash\{0\},\ V_1 = Z(R)^{\star}\backslash \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\}$.

These subsets are clearly nonempty and each contains nonadjacent vertices.

Moreover, $Z(R)^{*} = V_1 \bigcup \left\{\bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\}\right\}$.

Now $\mid \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\} \mid = p^{(\frac{k}{2}+h)r} - 1$ and $\mid V_1 \mid = 1$. So $Z(R)^{\star} = p^{(\frac{k}{2}+h)r}$ which implies that $\Gamma(R)$ is $p^{(\frac{k}{2}+h)r} -$ partite if k is even.

Case II: k is an odd integer.

Let $X_{i,s} = \{\Sigma_{i=1}^{r} a_i\lambda_i + \Sigma_{s=1}^{h}\lambda_s u_s\}$ where $a_i \in \{0, (j-1)p^{\frac{k+1}{2}}\}$ and $2 \leq j \leq p^{\frac{k-1}{2}}$. Then $Z(R)^{\star}$ is partitioned into the following mutually disjoint subsets,

$V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s} = X_{i,s}\backslash\{0\},\quad V_1 = Z(R)^{\star}\backslash \bigcup_{i,s}\ \{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\}$.

The subsets are nonempty and each contains nonadjacent vertices. So $Z(R)^{\star} = V_1 \bigcup \left\{\bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\}\right\}$. Now $\mid \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\}| = p^{(\frac{k-1}{2}+h)r} - 1$, $|V_1| = 1$. So $|Z(R)^{\star}| = |V_1| + \mid \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i+\Sigma\lambda_s u_s}\} \mid = p^{(\frac{k-1+2h}{2})r}$ showing that $\Gamma(R)$ is $p^{(\frac{k-1+2h}{2})r} -$ partite if k is odd. □

Corollary 4.4.8 *Let R be the ring in Construction I described in Proposition 4.4.7 and let $k \geq 3$ then;*

(i) $diam(\Gamma(R)) = 2$.

(ii) $gr(\Gamma(R)) = 3$.

$$(iii)\ b(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{k}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k}{2}+h)r}}, & if \quad k \ is \ even; \\ \frac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}}, & if \quad k \ is \ odd. \end{cases}$$

Proof.

(i) The annihilator, $ann_{R_0/pR_0}(Z(R)) = Z(R)^{k-1} \oplus U$. So the zero divisor of the form $p^{k-1}r_0 + \Sigma\lambda_i u_i$ is adjacent to every other nonzero zero divisor. Meanwhile, when $l + t \neq 0(\text{mod } k)$, $r_0, \ s_0 \in R_0$, then zero divisors of the form $p^l r_0 + \Sigma_{i=1}^{h}\lambda_i u_i$ and $p^t s_0 + \Sigma_{i=1}^{h}\lambda_i' u_i'$ are non adjacent. Thus $diam(\Gamma(R)) = 2$.

(ii) The order of $ann_{R_0/pR_0}(Z(R)) = p^{(h+1)r}$. So every zero divisor graph of R contains a complete subgraph, $K_{p^{(h+1)r}}$. The in variants reveal that the least polygon in $\Gamma(R)$ is K_3 and so the result follows.

(iii) Let k be an even integer and partition of $Z(R)^\star$ be as in proof of Proposition 4.4.7. Then $V_1 = Z(R)^\star \setminus \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i + \Sigma\lambda_s u_s}\}$, and
$N(V_1) = \bigcup_{i,s}\{V_{\Sigma a_i\lambda_i} + \Sigma\lambda_s u_s\}$ leading to $|N(V_1)| = p^{(\frac{k}{2}+h)r} - 1$.
Since $|V_1| = |Z(R)^\star| - |\bigcup_{i,s}\{V_{\Sigma a_i\lambda_i + \Sigma\lambda_s u_s}\}| = p^{(k-1+h)r} - 1 - \left(p^{(\frac{k}{2}+h)r} - 1\right)$
$= p^{(k-1+h)r} - p^{(\frac{k}{2}+h)r}$, the result follows immediately.
If k is odd, then $|N(V_1)| = |\bigcup_{i,s}\{V_{\Sigma a_i\lambda_i + \Sigma\lambda_s u_s}\}| = p^{(\frac{k-1}{2}+h)r} - 1$, and
$|V_1| = |Z(R)^\star - X_{i,s}| = |Z(R)^\star| - |X_{i,s}| = p^{(k-1+h)r} - 1 - \left(p^{(\frac{k-1}{2}+h)r} - 1\right)$
$= p^{(k-1+h)r} - p^{(\frac{k-1}{2}+h)r}$. Hence $\frac{|N(V_1)|}{|V_1|} = \frac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}}$.

□

Proposition 4.4.9 *Let R be a ring in the Construction I. Then*

$$\Gamma_E(R) = \begin{cases} \frac{k}{2} - \textit{partite}, & \textit{if } k \textit{ is even;} \\ \\ \frac{k+1}{2} - \textit{partite}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof.

If $k = 1$ or 2, then $\Gamma_E(R)$ is 1 - partite, so the result trivially holds. For $k \geq 3$, we consider the following cases.

Case I: k is even.

Let $\tilde{J^i} = J^i \oplus U, i \in \mathbb{N}$. Then the vertex set of $\Gamma_E(R)$ is partitioned into the following subsets; $V_1 = \{\tilde{J^l}\}$ for $1 \leq l \leq \frac{k}{2}$ and $V_j = \{\tilde{J^j}\}$ where $\frac{k}{2} < j \leq k-1$. For each $j, V_1 \cap V_j = \emptyset$ and V_j are mutually disjoint. Moreover $V_1 \bigcup \{\cup_{j=\frac{k}{2}+1}^{k-1} V_j\} = \Gamma_E(R)$. The result follows by counting the disjoint subsets.

Case II: k is odd.

The set of vertices of $\Gamma_E(R)$ are partitioned into the following subsets; $V_1 = \{\tilde{J^l}\}$ for $1 \leq l \leq \frac{k-1}{2}$ and $V_j = \{\tilde{J^j}\}$ for $\frac{k-1}{2} < j < k-1$. Then clearly as in Case I above, for each j, $V_1 \cap V_j = \emptyset$ and V_j are mutually disjoint. Moreover, $V_1 \bigcup \{\cup_{j=\frac{k-1}{2}+1}^{k-1} V_j\} = \Gamma_E(R)$. The result follows by counting the disjoint subsets. □

Chapter Summary

In the sequel, the remarks and subsequent propositions summarize the characterization of the zero divisor graphs of completely primary finite rings given in the Construction I.

Remark 4.4.10 *Our results show that if R is a commutative ring in the Construction I, then its Jacobson radical $J(R)$ is equal to its subset of zero divisors, $Z(R)$. Thus $Z(R)$ is a nilpotent ideal implies that if R is not a field, then $ann(Z(R)) \neq (0)$. Moreover, element of $ann(Z(R)^\star)$ are adjacent vertices in the graph $\Gamma(R)$.*

Remark 4.4.11 *Let R_1 and R_2 be commutative rings in Construction I, then*

$\Gamma(R_1) \cong \Gamma(R_2)$ *does not imply that* $R_1 \cong R_2$. *For instance, if* $R_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* $R_2 = \mathbb{Z}_4 \oplus \mathbb{Z}_2$, *then* $\Gamma(R_1)$ *and* $\Gamma(R_2)$ *are triangular, while* $R_1 \ncong R_2$.

Proposition 4.4.12 *There exist three non-isomorphic commutative rings in Construction I whose zero divisor graphs are triangular.*

Proof.

See Corollary 4.3.2 □

Proposition 4.4.13 *There exists no ring in Construction I whose zero divisor graph,* $\Gamma(R)$ *is an* $n - gon,\ n > 3$.

Proof.

Suppose R is not a field and the cardinality of the vertices of nonzero zero divisors, $|V(\Gamma(R))| > 3$. Since $Z(R)$ is a nilpotent ideal, $ann(Z(R)) \neq (0)$. Consider $0 \neq a \in ann(Z(R))$, then a is adjacent to every other $b \in V(\Gamma(R))$. This completes the proof. □

Proposition 4.4.14 *Let* R *be a commutative ring in Construction I. Then* $diam\,(\Gamma(R)) = 0,\ 1$ *or* 2.

Proof.

If $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $|Z(R)^{\star}| = 1;$ So $diam\,(\Gamma(R)) = 0$.

Let $char R = p$ or p^2, then $\Gamma(R)$ is complete so that $diam(\Gamma(R)) = 1$ (with the exception of the case when $p = 2$ and $h = 1$).

For all the other commutative rings considered under this construction, $diam\,(\Gamma(R)) = 2$ because $0 \neq a \in ann\,(Z(R))$ is adjacent to every other element in $Z(R)^{\star}$ and the proof is similar to part (i) of Corollary 4.4.8. □

Proposition 4.4.15 *Let R be a commutative ring in Construction I. Then the girth $gr\,(\Gamma(R)) = \infty$ if R is one of the following rings.*

(i) $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(ii) $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Proof.

In case (i), there exists only one non zero divisor, that is $(0, 1)$.

In case (ii), there exists two nonzero zero divisors, $(0, 1)$ and $(0, 2)$. □

Theorem 4.4.16 *Let R be a commutative ring that is not a field, described by Construction I. Then,*

$$(i)\quad \Gamma(R) = \begin{cases} K_{p^{rh}-1}, & \textit{if } k = 1; \\ K_{p^{(h+1)r}-1}, & \textit{if } k = 2; \\ p^{(\frac{k}{2}+h)r} - \textit{partite}, & \textit{if } k \geq 4 \textit{ is even}; \\ p^{(\frac{k-1+2h}{2})r} - \textit{partite}, & \textit{if } k \geq 3 \textit{ is odd}. \end{cases}$$

$$(ii)\quad diam\,(\Gamma(R)) = \begin{cases} 0, & \textit{if } p = 2,\ h = 1; \\ 1, & \textit{if } k = 1,\ 2; \\ 2, & \textit{elsewhere}. \end{cases}$$

$$(iii)\quad gr\,(\Gamma(R)) = \begin{cases} \infty, & \textit{if } r = 1,\ h = 1, \textit{and } p = 2,\ 3; \\ 3, & \textit{elsewhere}. \end{cases}$$

$$\textit{(iv)}\quad b\left(\Gamma(R)\right)=\begin{cases}\frac{p^{(\frac{k}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k}{2}+h)r}}, & k\geq 4 \ \textit{and even};\\ \frac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}}, & k\geq 3 \ \ \textit{and odd};\\ \infty, & k=1,2.\end{cases}$$

PROOF.

Follows from Propositions 4.2.1, 4.3.1, 4.4.4, 4.4.7 and Corollary 4.4.8. □

Theorem 4.4.17 *Let R be a ring in the Construction I. Then*

$$\Gamma_E\left(R\right)=\begin{cases}\frac{k}{2}\ -\ \textit{partite}, & \textit{if}\ \ k\ \textit{is even};\\ \frac{k+1}{2}\ -\ \textit{partite}, & \textit{if}\ \ k\ \ \textit{is odd}.\end{cases}$$

PROOF.

Follows from Proposition 4.4.9 .

□

Chapter 5

A class of finite rings II

Let r be a positive integer and $2 \leq k \in \mathbb{Z}$.. Let $GR(p^{kr},\ p^k)$ be a Galois ring of order p^{kr} and characteristic p^k. Consider $R = GR(p^{kr},\ p^k) \oplus U$ where U is a finitely generated $GR(p^{kr},\ p^k)$- module. The structure of unit groups of R have been extensively studied (See [24]). The set, $Z(R)$ of zero divisors of R satisfy the condition $(Z(R))^2 \subseteq GR(p^{kr},\ p^k)$ and R is well known (See [16])to be completely primary finite ring. In this chapter, the structure of zero divisors graphs of R have been investigated.

5.1 Construction II

Let r and k be a positive integers with $k \geq 2$. Let $R_0 = GR(p^{kr},\ p^k)$ be a Galois ring. For each $i = 1, \ldots, h$, let $u_i \in Z(R)$ such that U is an $h-$ dimensional R_0 module so that $R = R_0 \oplus U$ is an additive Abelian group. On R define multiplication as follows;

For $r_0, s_0 \in R_0, \alpha_i, \omega_i, \lambda_{ij} \in R_0/pR_0$ and $\sigma_i \in Aut(R_0)$, let $\left(r_0+\Sigma_{i=1}^{h}\alpha_i u_i\right)\left(s_0+\Sigma_{i=1}^{h}\omega_i u_i\right)$

$= r_0s_0+p^{k-1}\Sigma_{i,j=1}^{h}\lambda_{ij}\left(\alpha_i(\omega_j)^{\sigma_i}+pR_0\right)+\Sigma_{i=1}^{h}\left[(r_0+pR_0)\omega_i+\alpha_i(s_0+pR_0)^{\sigma_i}\right]u_i$. We verify that this multiplication turns R into a ring with identity $(1,0,0,\ldots,0)$. Since $R_0\oplus U$ is an additive Abelian group, we show that it is a multiplicative semigroup in which multiplication distributes over addition.

Now, $\left(r_0+\Sigma\lambda_i u_i\right)\left(\left(s_0+\Sigma\gamma_i u_i\right)\left(t_0+\Sigma\kappa_i u_i\right)\right)$

$= \left(r_0+\Sigma\lambda_i u_i\right)\left(s_0t_0+p^{k-1}\Sigma\beta_{ij}\left(\gamma_i\left(\kappa_j\right)^{\sigma_i}+pR_0\right)+\Sigma\left[\left(s_0+pR_0\right)\kappa_i+\gamma_i\left(t_0+pR_0\right)^{\sigma_i}\right]u_i\right)$

$= r_0s_0t_0+r_0p^{k-1}\Sigma\beta_{ij}\left(\gamma_i\left(\kappa_j\right)^{\sigma_i}+pR_0\right)+\Sigma\left(r_0+pR_0\right)\left(\left(s_0+pR_0\right)\kappa_i+\gamma_i\left(t_0+pR_0\right)^{\sigma_i}\right)+$

$\lambda_i\left(s_0t_0+p^{k-1}\Sigma\beta_{ij}\left(\gamma_i\left(\kappa_j\right)^{\sigma_i}\right)+pR_0\right)^{\sigma_i}u_i$

$= r_0s_0t_0+r_0p^{k-1}\Sigma\beta_{ij}\left(\gamma_i\left(\kappa_j\right)^{\sigma_i}+pR_0\right)+$

$\left(\Sigma\left(r_0s_0+pR_0\right)\kappa_i+\left(\left(r_0+pR_0\right)\gamma_i+\lambda_i\left(s_0+pR_0\right)\right)+p^{n-1}\Sigma\beta_{ij}\left(\gamma_i\left(\kappa_j\right)^{\sigma_i}+pR_0\right)^{\sigma_i}\left(t_0+pR_0\right)^{\sigma_i}\right)u_i$

$= \left(\left(r_0s_0+p^{k-1}\Sigma\phi_{ij}\lambda_i\left(\gamma_j\right)^{\sigma_i}+pR_0\right)+\left(\left(r_0+pR_0\right)\gamma_i+\lambda_i\left(s_0+pR_0\right)^{\sigma_i}\right)\right)u_i\left(t_0+\Sigma\kappa_i\right)$

(where $\phi_{ij}=r_0t_0\beta_{ij}$)

$= \left(\left(r_0+\Sigma\lambda_i u_i\right)\left(s_0+\Sigma\gamma_i u_i\right)\right)\left(t_0+\Sigma\kappa_i u_i\right).$

Next,

$\left(r_0+\Sigma\lambda_i u_i\right)\left(\left(s_0+\Sigma\gamma_i u_i\right)+\left(t_0+\Sigma\kappa_i u_i\right)\right)=\left(r_0+\Sigma\lambda_i u_i\right)\left(\left(s_0+t_0\right)+\Sigma\left(\gamma_i+\kappa_i\right)u_i\right)$

$= r_0\left(s_0+t_0\right)+p^{k-1}\Sigma\beta_{ij}\left(\lambda_i\left(\gamma_j+\kappa_j\right)^{\sigma_i}+\Sigma\left(\left(r_0+pR_0\right)\left(\gamma_i+\kappa_i\right)+\lambda_i\left(s_0+t_0\right)+pR_0\right)^{\sigma_i}\right)u_i$

$= r_0s_0+p^{k-1}\Sigma\beta_{ij}\left(\lambda_i\left(\gamma_j\right)^{\sigma_i}+\Sigma\left(r_0+pR_0\right)\gamma_i+\lambda_i\left(s_0+pR_0\right)^{\sigma_i}\right)u_i+r_0t_0$

$+\,p^{k-1}\Sigma\beta_{ij}\left(\lambda_i\left(\kappa_j\right)^{\sigma_i}+\Sigma\left(r_0+pR_0\right)\kappa_i+\lambda_i\left(t_0+pR_0\right)^{\sigma_i}\right)u_i$

$= \left(r_0+\Sigma\lambda_i u_i\right)\left(s_0+\Sigma\gamma_i u_i\right)+\left(r_0+\Sigma\lambda_i u_i\right)\left(t_0+\Sigma\kappa_i u_i\right).$

Similarly it can be shown that

$$\left(\left(r_0+\Sigma\lambda_i u_i\right)+\left(s_0+\Sigma\gamma_i u_i\right)\right)\left(t_0+\Sigma\kappa_i u_i\right)=\left(r_0+\Sigma\lambda_i u_i\right)\left(t_0+\Sigma\kappa_i u_i\right)+\left(s_0+\Sigma\gamma_i u_i\right)\left(t_0+\Sigma\kappa_i u_i\right).$$

Now, suppose $\left(\left(r_0+\Sigma\alpha_i u_i\right)\left(s_0+\Sigma\lambda_i u_i\right)\right)=r_0+\Sigma\alpha_i u_i$. Then

$r_0 s_0 + p^{k-1}\Sigma\beta_{ij}\left(\alpha\left(\lambda_j\right)^{\sigma_i} + pR_0\right) = r_0$ and $\left(r_0 + pR_0\right)\lambda_i + \alpha_i\left(s_0 + pR_0\right)^{\sigma_i} = \alpha_i$. So $\left(r_0 + pR_0\right)\lambda_i = 0$ implies that $\lambda_i = 0$ and $\left(s_0 + pR_0\right)^{\sigma_i} = 1$. Since σ_i is an automorphism $s_0 + pR_0 = 1 + pR_0$ which means that $s_0^{-1} \in pR_0$ so that s_0 is a unit in R_0. Clearly $r_0 s_0 = r_0$ means that $s_0 = 1$.

Lemma 5.1.1 *Let R be a ring in Construction II. R is commutative if and only if $\sigma_i = id_{R_0}$, (the identity automorphism) for every $i = 1, \ldots, h$.*

Proof.

If $\sigma_i = id_{R_0}$, then commutativity of R follows from the definition of multiplication. Conversely, let R be commutative. Then for each $a_0, b_0 \in R, \alpha_i, \beta_i \in R_0/pR_0$, $\left(a_0 + \Sigma\alpha_i u_i\right)\left(b_0 + \Sigma\beta_i u_i\right) = \left(b_0 + \Sigma\beta_i u_i\right)\left(a_0 + \Sigma\alpha_i u_i\right)$. This implies that $a_0 b_0 + p^{k-1}\Sigma\beta_{ij}\left(a_i\left(b_j\right)^{\sigma_i} + pR_0\right) + \Sigma\left(\left(a_0 + pR_0\right)\beta_i + \alpha_i\left(b_0 + pR_0\right)^{\sigma_i}\right)u_i$ $= b_0 a_0 + p^{k-1}\Sigma\beta_{ij}\left(b_i\left(a_j\right)^{\sigma_i} + pR_0\right) + \Sigma\left(\left(b_0 + pR_0\right)\alpha_i + \beta_i\left(a_0 + pR_0\right)^{\sigma_i}\right)u_i$ which impies that $\sigma_i = id_{R_0}$. □

Proposition 5.1.2 *Let $k = 2$, then R is a completely primary finite ring of characteristic p^2 satisfying:*

(i) $Z(R) = pR_0 \oplus U$.

(ii) $\left(Z\left(R\right)\right)^2 = pR_0$.

(iii) $\left(Z\left(R\right)\right)^3 = (0)$.

Proof.

Similar to the proof of Proposition 4.4.3 with some slight modification.

□

Proposition 5.1.3 *Let $R_0 = GR\left(p^{2r},\ p^2\right)$ be a Galois ring. For each $i = 1, \ldots, h$, let $u_i \in Z(R)$ such that U is an $h-$dimensional R_0 module so that $R = R_0 \oplus U$ is an additive Abelian group. On R, define multiplication as follows:*

$$(r_0 + \Sigma\alpha_i u_i)(s_0 + \Sigma\omega_i u_i) = r_0 s_0 + p\Sigma\lambda_{ij}\left(\alpha_i(\omega_j)^{\sigma_i} + pR_0\right) + \Sigma_i^h\left[(r_0 + pR_0)\,\omega_i + \alpha_i(s_0 + pR_0)^{\sigma_i}\right]u_i$$

where $r_0, s_0 \in GR\left(p^{2r},\ p^2\right), \alpha_i, \omega_i, \lambda_{ij} \in R_0/pR_0$ and $\sigma_i \in Aut\,(R_0)$. The graph of R, $\Gamma(R)$, satisfies the following:

(i) $\mid(\Gamma(R))\mid = p^{(h+1)r} - 1.$

(ii) $diam\,(\Gamma(R)) = 2.$

(iii) $gr\,(\Gamma(R)) = \begin{cases} \infty, & r = 1, h = 1, p = 2; \\ 3, & elsewhere. \end{cases}$

(iv) The binding number, $b\,(\Gamma(R)) = \frac{p^r - 1}{p^{(h+1)r} - p^r}.$

Proof.

(i) Clearly $Z(R) = pR_0 \oplus U$. Since $Z(R)$ is a maximal ideal of R, the quotient $R/Z(R)$ is a field of order p^r. Now consider $0 \neq a \in R/Z(R)$ then, $(R/Z(R))^\star = < a >$ and $o(a) = p^r - 1$. This shows that each element which does not belong to $Z(R)$ has an inverse. Thus $\mid Z(R)\mid = p^{(h+1)r}$ and $\mid Z(R)^\star\mid = p^{(h+1)r} - 1$.

(ii) The annihilator, $ann\,(Z(R)) = pR_0$. Now, let $x \notin pR_0$, then there exists $y \in Z(R)^\star$ such that $xy \in pR_0$. But $xyz = 0$, where $z \in ann\,(Z(R)) = pR_0$. So $diam\,(\Gamma(R)) = 2$.

(iii) If $r = 1, p = 2, h = 1$, then $Z(R)^\star = \{(0,1), (2,0), (2,1)\}$. In this case $\Gamma(R)$ is a bipartite graph, since $(2,0)$ is adjacent to the other two vertices while $(0,1)$ and $(2,1)$ are non adjacent. Elsewhere, $\mid(ann\,(Z(R)))^\star\mid = p^r \geq 2$.

Now, let $x, y \in (ann\,(Z\,(R)))^{\star}$, then x and y are adjacent. Moreover, any $z \in Z\,(R)^{\star}$ is adjacent to x and y. This completes the proof.

(iv) Consider $N(S) = ann\,(Z(R))^{\star} = pR_0 \backslash \{0\}$. So $\mid N(S) \mid = p^r - 1$. Now, $S = V\,(\Gamma\,(R))\,\backslash N\,(S)$, so that $\mid S \mid = p^{(h+1)r} - 1 - (p^r - 1)$. Thus $b\,(\Gamma(R)) = \frac{\mid N(S) \mid}{\mid S \mid} = \frac{p^r - 1}{p^{(h+1)r} - p^r}$.

□

Proposition 5.1.4 *Let R be a ring in Construction II with $h = 1$, then*

$$\Gamma\,(R) = \begin{cases} p^{\left(\frac{k}{2}\right)r} \ - partite, & if \ \ k \ is \ even \ ; \\ p^{\left(\frac{k-1}{2}\right)r} \ - partite, & if \, k \ \ is \ odd \ . \end{cases}$$

Proof.

Consider $\lambda_1, \ldots, \lambda_r \in R_0$ with $\lambda_1 = 1$ such that $\lambda_1, \ldots, \lambda_r \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Since the two cases do not overlap, we treat them in turn.

Case I: k is an even integer.

Let $X = \left\{\Sigma_{i=1}^{r} a_i \lambda_i\right\}$, $a_i \in \left\{0, j\left(p^{\frac{k}{2}}\right)\right\}, 1 \leq j \leq p^{\frac{k}{2}} - 1$. Then, $Z(R)^{\star}$ is partitioned into the following subsets;

$\bigcup V_{\Sigma_i a_i \lambda_i} = X \backslash \{0\}, V_1 = Z(R)^{\star} \backslash \bigcup V_{\Sigma_i a_i \lambda_i}$ and $Z(R)^{\star} = V_1 \bigcup (\bigcup V_{\Sigma_i a_i \lambda_i})$.

The subsets are nonempty, mutually disjoint and contain nonadjacent vertices. Moreover, $\left| \bigcup V_{\Sigma_i a_i \lambda_i} \right| = p^{\left(\frac{k}{2}\right)r} - 1$ so that $\Gamma\,(R)$ is $p^{\left(\frac{k}{2}\right)r}$- partite.

Case II: k is an odd integer.

Let $X = \left\{\Sigma_i^r a_i \lambda_i\right\}$, $a_i \in \left\{0, (j-1)\,p^{\frac{k+1}{2}}\right\}, 2 \leq j \leq p^{\frac{k-1}{2}}$. Partition $Z\,(R)^{\star}$ into the following mutually disjoint subsets; $\bigcup V_{\Sigma_i a_i \lambda_i} = X \backslash \{0\}, V_1 = Z(R)^{\star} \backslash \bigcup V_{\Sigma_i a_i \lambda_i}$.

Then subsets are nonempty and each contains nonadjacent vertices. In addition, $\left|\bigcup V_{\Sigma_i a_i \lambda_i}\right| = p^{(\frac{k-1}{2})r} - 1$ so that $\Gamma(R)$ is $p^{(\frac{k-1}{2})r}$- partite.

□

Example 5.1.5 *Consider the ring $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4/2\mathbb{Z}_4$ with respect to multiplication in Construction II. The set $Z(R)^\star$ of nonzero zero divisors of R is*

$Z(R)^\star = \{(0,1), (2,0), (2,1)\}$ and the corresponding zero divisor graph is illustrated in Figure 5.1 below. This is a bipartite graph with $deg((0,1)) = deg((2,1)) = 1$ while $deg((2,0)) = 2$. Moreover, $diam(\Gamma(R)) = 2, gr(\Gamma(R)) = \infty$ and $b(\Gamma(R)) = \frac{1}{2}$.

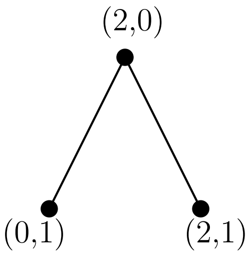


Figure 5.1: The zero divisor graph of $\mathbb{Z}_4 \oplus \mathbb{Z}_4/2\mathbb{Z}_4$

Proposition 5.1.6 *Let $k \geq 3$, then R is a completely primary finite ring of characteristic p^k, satisfying;*

(i) $Z(R) = pR_0 \oplus U$.

(ii) $(Z(R))^{k-1} = p^{k-1}R_0$.

(iii) $(Z(R))^k = (0)$.

Proof.

Similar to the proof of Proposition 4.4.3. □

Proposition 5.1.7 *Let $k \geq 3$ and R be a ring in Construction II, then the graph $\Gamma(R)$ satisfies the following;*

(i) $\mid (\Gamma(R)) \mid = p^{(h+k-1)r} - 1.$

(ii) $diam\,(\Gamma(R)) = 2.$

(iii) $gr\,(\Gamma(R)) = 3.$

(iv) $$b\,(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{k}{2})r}-1}{p^{(k-1+h)r}-p^{(\frac{k}{2})r}}, & \textit{if } k \ \ \textit{is} \ \ \textit{even} \ ; \\ \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2})r}}, & \textit{if } k \ \ \textit{is} \ \ \textit{odd.} \end{cases}$$

Proof.

For part (i), line of proof is similar to part (i) of Proposition 5.1.3 except that now $|Z(R)| = p^{(h+k-1)r}$ and hence $|Z(R)^{\star}| = p^{(h+k-1)r} - 1$.

For parts (ii) and (iii)the proofs are similar to Proposition 5.1.3.

For part (iv), we consider the two cases separately.

Case I: k is even.

Let $X = \{\Sigma_{i=1}^{r} a_i\lambda_i\}$, $a_i \in \left\{0, j\left(p^{\frac{k}{2}}\right)\right\}, 1 \leq j \leq p^{\frac{k}{2}} - 1$, then define $V_{\Sigma_i a_i\lambda_i} = X \backslash \{0\}$ and $V_1 = Z(R)^{\star} \backslash \bigcup V_{\Sigma_i a_i\lambda_i}$. From the definition of V_1,

$N(V_1) = \bigcup V_{\Sigma_i a_i\lambda_i}$. So $\mid N(V_1) \mid = p^{(\frac{k}{2})r} - 1$. Also $|V_1| = \mid Z(R)^{\star} - N(V_1)|$

$= \mid Z(R)^{\star}| - | \bigcup V_{\Sigma_i a_i\lambda_i} \mid = p^{(k-1+h)r} - 1 - (p^{(\frac{k}{2})r} - 1) = p^{(k-1+h)r} - p^{(\frac{k}{2})r}$. The binding number is then established by the ratio $\frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{(\frac{k}{2})r}-1}{p^{(k-1+h)r}-p^{(\frac{k}{2})r}}$.

Case II: k is odd.

Let $X = \{\Sigma_{i=1}^{r} a_i\lambda_i\}$, $a_i \in \left\{0, (j-1)\,p^{\frac{k+1}{2}}\right\}, 2 \leq j \leq p^{\frac{k-1}{2}}$. Then define $V_{\Sigma_i a_i\lambda_i} = X \backslash \{0\}$ and $V_1 = Z(R)^{\star} \backslash \bigcup_i V_{\Sigma_i a_i\lambda_i}$. From the definition of V_1,

$N(V_1) = \bigcup V_{\Sigma_i a_i\lambda_i}$, so that $\mid N(V_1) \mid = p^{(\frac{k-1}{2})r} - 1$. Also, $\mid V_1 \mid = \mid Z(R)^{\star} - N(V_1)|$

$= \mid Z(R)^{\star} \mid - \mid \bigcup V_{\Sigma_i a_i \lambda_i} \mid = p^{(k-1+h)r} - 1 - (p^{(\frac{k-1}{2})r} - 1) = p^{(k-1+h)r} - p^{(\frac{k-1}{2})r}$. The result follows from the ratio $\frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{(\frac{k-1}{2})r} - 1}{p^{(k-1+h)r} - p^{(\frac{k-1}{2})r}}$. □

Example 5.1.8 *Let $R_0 = GR(2^4,\ 2^2) \cong \mathbb{Z}_4[x]/(x^2+1)$, so that $k = 2$ and $r = 2$. Let $R = R_0 \oplus R_0/pR_0$ and let α be the root of x^2+1 in $\mathbb{Z}_4$. Then with respect to multiplication in Construction II, the set of nonzero zero divisors is*

$Z(R)^{\star} = \{(0,1), (2,0), (2,1), (2\alpha,0), (2\alpha,1), (2\alpha+2,0), (0,\alpha), (0,\alpha+1), (2,\alpha), (2,\alpha+1), (2\alpha+2,\alpha), (2\alpha+2,\alpha+1), (2\alpha+2,1)\}$

$X = \{\Sigma_i^r a_i \lambda_i\}$ So $i = 1$or 2 and $a_i \in \{0, j(p^{\frac{k}{2}})\}$ and $1 \leq j \leq p^{\frac{k}{2}} - 1$.Thus $j = 1$ so $a_i \in \{0,2\}$. Moreover, $\lambda_i \in \{1,\alpha\}$. Hence $X = \{0,\ \ 2,\ 2\alpha,\ \ 2\alpha+2\}$. giving $V_0 = (0,0)$, $V_2 = (2,0), V_{2\alpha} = (2\alpha,0)$ and $V_{2\alpha+2} = (2\alpha+2,0)$ so that $\bigcup V_{\Sigma_i^r a_i \lambda_i} = X \backslash \{0\} = \{V_2 = (2,0), V_{2\alpha} = (2\alpha,0),\ V_{2\alpha+2} = (2\alpha+2,0)\}$

The zero divisor graph of R is given in Figure 5.2 below.

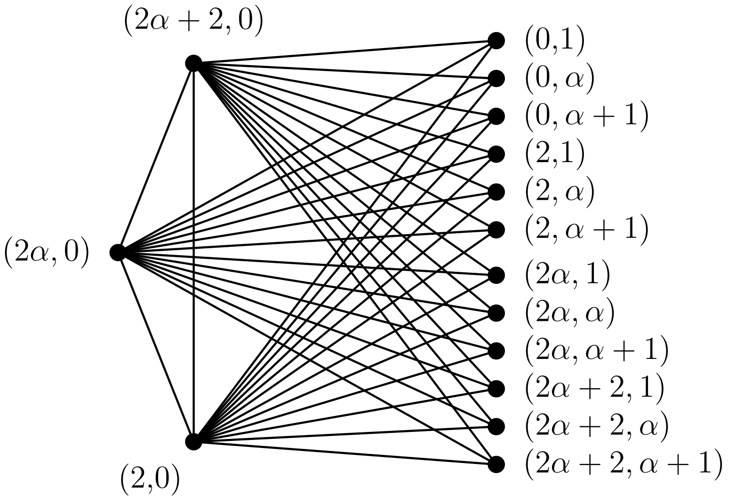


Figure 5.2: The zero divisor graph of $R = R_0 \oplus R_0/pR_0$ where $R_0 = GR(16,\ 4)$

The graph is 4-partite in which all the vertices are of degree 3 except for the *vertices* $(2,0),(2\alpha,0)$ *and* $(2\alpha+2,0)$ *which are each of degree 12. The diameter of the graph is 2 while its binding number is* $\frac{1}{4}$ *and the girth is 3.*

Corollary 5.1.9 *Let* R *be a ring in Construction II. Then the clique number of* $\Gamma(R)$ *is given by*

$$\omega(\Gamma(R)) = \begin{cases} p^{(\frac{k}{2})r}, & \text{if } k \text{ is even;} \\ \\ p^{(\frac{k-1}{2})r}, & \text{if } k \text{ is odd .} \end{cases}$$

Proof.

The clique number coincides with the number of partite subsets since each subset of vertices has at least a vertex which is adjacent to another vertex in a distinct subset.

□

Proposition 5.1.10 *Let* R *be a ring in Construction II. Then*

$$\Gamma_E(R) = \begin{cases} \frac{k}{2} - \text{ partite}, & \text{if } k \text{ is even;} \\ \\ \frac{k+1}{2} - \text{ partite}, & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

Case I: k is even. $\Gamma_E(R)$ is partitioned into the following subsets:

$V_1 = \{(Z(R))^i\}, \ \ 1 \leq \ i \ \leq \frac{k}{2}$.

$V_j = \{(Z(R))^j\}$ such that $\frac{k}{2} \ < \ j \leq k-1$.

For each j, we have $V_1 \cap V_j = \emptyset$ and V_j are mutually disjoint for all j.

Moreover, $V_1 \bigcup \{\bigcup_{j=\frac{k}{2}+1}^{k-1} V_j\} = V(\Gamma_E(R))$.

The result follows by counting the disjoint subsets.

Case II: k is odd.

We partition $\Gamma_E(R)$ in to the following subsets:

$V_1 = \{(Z(R))^i\},\ \ 1 \leq\ i\ \leq \frac{k-1}{2},$

$V_j = \{(Z(R))^j\},\ \frac{k-1}{2}\ <\ j \leq k-1$. For each $j,\ V_1 \cap V_j = \emptyset$ and V_j are mutually disjoint. Furthermore $V_1 \bigcup \{\bigcup_{j=\frac{k-1}{2}}^{k-1} V_j\} = \Gamma_E(R)$. By counting the disjoint subsets, we obtain the result.

□

As a consequence to the immediate proposition, we have the following result.

Corollary 5.1.11 *Let $\Gamma_E(R)$, be the graph determined by the equivalence classes of the zero divisors of a commutative ring given by Construction II. Then, the clique number of the graph is given by*

$$\omega(\Gamma_E(R)) = \begin{cases} \frac{k}{2}, & \text{if } k \text{ is even };\\ \frac{k+1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 5.1.12 *Let R be a ring in the Construction II. Then*

$$\text{(i) } diam(\Gamma_E(R)) = \begin{cases} 0, & \text{when } k = 2;\\ 1, & \text{when } k = 3;\\ 2, & \text{elsewhere.} \end{cases}$$

$$\text{(ii) } gr(\Gamma_E(R)) = \begin{cases} \infty, & \text{if } k = 2,\ 3;\\ 3, & \text{if } k > 3. \end{cases}$$

$$\textit{(iii)}\ b(\Gamma_E(R)) = \begin{cases} 0, & \textit{if } k = 2; \\ 1, & \textit{if } k = 3; \\ \frac{k-4}{k}, & \textit{if } k \geq 4 \textit{ and is even }; \\ \frac{k-5}{k-1}, & \textit{if } k \geq 4 \textit{ and is odd }. \end{cases}$$

Proof.

(i) If $k = 2$, then $\Gamma_E(R))$ has a single vertex. If $k = 3$, $\Gamma_E(R))$ is a line graph

$$[(Z(R)]\rule[0.5ex]{6em}{0.4pt}[(Z(R))^2]$$

Now, for $k > 3$, $[(Z(R))^{k-1}]$ is adjacent to every other vertex in $\Gamma_E(R))$. But $\Gamma_E(R))$ is not complete because, if $0 < i < \frac{k}{2}$ where k is even and $0 < i < \frac{k-1}{2}$, where k is odd, there exists $j > i > 0$ so that $[(Z(R))^i][(Z(R))^{k-1-j}] = [(Z(R))^{k-1+i-j}] \neq 0$.

(ii) For $k = 2$ or 3, the result follows from (i). Elsewhere,

$$[(Z(R))^{k-1}]\rule[0.5ex]{5em}{0.4pt}[(Z(R)^i]\rule[0.5ex]{5em}{0.4pt}[(Z(R))^{k-i}]\rule[0.5ex]{5em}{0.4pt}[(Z(R))^{k-1}]$$

is a cycle, if $0 < i < \frac{k}{2}$ when k is even or when $0 < i < \frac{k-1}{2}$ and k is odd. It is important to note that there exists no $n - gon, n > 3$ because $[(Z(R))^{k-1}]$ is adjacent to every other vertex in $\Gamma_E(R)$.

(iii) If $k = 2$, then $S = Z(R)$ and $N(S) = \emptyset$. So $\mid S \mid = 1$ and $\mid N(S) \mid = 0$. If $k = 3$ then $\big| N(S) \big| = 1$. Now, let $k > 3$. By Corollary 3.2.8, when k is even $\big| V_1 \big| = \frac{k}{2}$ while $\big| N(V_1) \big| = \frac{k-4}{2}$. When k is odd, then $\big| V_1 \big| = \frac{k-1}{2}$ while $\big| N(V_1) \big| = \frac{k-5}{2}$. Then by the ratio $\frac{\mid N(V_1) \mid}{\mid V_1 \mid}$ for the binding number, we obtain the result.

□

Chapter summary

In summary, the characterization of rings in the Construction II is as follows;

Theorem 5.1.13 *Let R be a commutative ring described by Construction II. Then the zero divisor graph of R satisfies the following*

(i) $\mid (\Gamma(R)) \mid = p^{(h+k-1)r} - 1$.

(ii) $diam\,(\Gamma(R)) = 2$.

(iii) $gr\,(\Gamma(R)) = 3$.

(iv) $$b\,(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{k}{2})r}-1}{p^{(k-1+h)r}-p^{(\frac{k}{2})r}}, & if\, k \quad is \quad even\ ; \\ \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2})r}}, & if\, k \quad is \quad odd. \end{cases}$$

Theorem 5.1.14 *Let R be a ring in Construction II with $h = 1$, then*

$$\Gamma\,(R) = \begin{cases} p^{\left(\frac{k}{2}\right)r} \ - partite, & if \ \ k \ is \ \ even\ ; \\ p^{\left(\frac{k-1}{2}\right)r} \ - partite, & if\, k \ \ is \ \ odd\ . \end{cases}$$

Theorem 5.1.15 *Let R be a ring in Construction II. Then the clique number of $\Gamma(R)$ is given by*

$$\omega(\Gamma(R)) = \begin{cases} p^{(\frac{k}{2})r}, & if \ \ k \ is \ \ even; \\ p^{(\frac{k-1}{2})r}, & if \ \ k \ is \ \ odd\ . \end{cases}$$

Theorem 5.1.16 *Let R be a ring in Construction II. Then*

$$\Gamma_E\,(R) = \begin{cases} \frac{k}{2} - \ partite, & if \ \ k \ \ is\ even; \\ \frac{k+1}{2} - \ partite, & if\, k\ is\ odd. \end{cases}$$

Corollary 5.1.17 *Let* $\Gamma_E(R)$*, be the graph determined by the equivalence classes of the zero divisors of a commutative ring given by Construction II. Then, the clique number of the graph is given by*

$$\omega(\Gamma_E(R)) = \begin{cases} \frac{k}{2}, & \text{if } k \text{ is even ;} \\ \frac{k+1}{2}, & \text{if } k \text{ is odd .} \end{cases}$$

Theorem 5.1.18 *Let* R *be a ring in the Construction II. Then*

(i) $$diam(\Gamma_E(R)) = \begin{cases} 0, & \text{when } k = 2; \\ 1, & \text{when } k = 3; \\ 2, & \text{elsewhere.} \end{cases}$$

(ii) $$gr(\Gamma_E(R)) = \begin{cases} \infty, & \text{if } k = 2,\ 3; \\ 3, & \text{if } k > 3. \end{cases}$$

(iii) $$b(\Gamma_E(R)) = \begin{cases} 0, & \text{if } k = 2; \\ 1, & \text{if } k = 3; \\ \frac{k-4}{k}, & \text{if } k \geq 4 \text{ and is even ;} \\ \frac{k-5}{k-1}, & \text{if } k \geq 4 \text{ and is odd .} \end{cases}$$

Chapter 6

Conclusion and Recommendation

In this chapter we conclude the thesis and provide some recommendations for further research.

Conclusion

In this study we have identified and investigated the zero divisor graphs of the Galois rings of the form $GR\left(p^{kr},\ p^{k}\right)$. In Proposition 3.1.2 we have not only identified the zero divisors of the trivial Galois rings, $GR\left(p^{k},\ p^{k}\right)$ but have further improved on the findings of Duane [19], (See a summary of this in Chapter 2 of this thesis) by establishing a general characterization of the graph $\Gamma(\mathbb{Z}_{p^k})$ for all k.

In as much as our findings are in agreement with those of Anderson and Livingston [7] on the diameter and the girth of the graph $\Gamma(\mathbb{Z}_{p^k})$, in Proposition 3.1.3, we have given a more robust characterization for the binding number of $\Gamma(\mathbb{Z}_{p^k})$ for all p, a prime number and for all $k \geq 3$. Sankeether, Sankar, Vasanthakumari and Meena [26] only generalized this for $\Gamma(\mathbb{Z}_{2^k})$.

What now follows is a summary of our results on both the Galois rings as well as on the classes of rings encountered in the the two ring constructions studied. For a commutative ring R which is not a field discussed in this thesis, the range of its diameter is $0 \leq diam(\Gamma(R)) \leq 2$, while its girth is either 3 or ∞. If S is another ring considered in the thesis, then $\Gamma(R) \cong \Gamma(S)$ does not imply that $R \cong S$. We also observed that there is no ring R considered in this thesis whose zero divisor graph $\Gamma(R)$, is an $n - gon$ where $n > 3$. Moreover, $diam\,(R_0) = diam\,(R_0 \oplus U) = 2$ if $R_0 = GR\left(p^{kr},\ p^k\right),\ k \geq 3$. If $1 \leq k \leq 2$, the diameter of R_0 is not preserved in the idealization. The thesis has provided partial answers to the following open questions.

- For any commutative finite ring R, it is known that $0 \leq diam\Gamma\,(R) \leq 3$. Which rings have zero divisor graphs of a specific diameter in this range?

- It is known that in any commutative finite ring R, either $3 < gr\,(\Gamma\,(R)) < 7$ or $gr\,(\Gamma\,(R)) = \infty$. It has been conjectured that there exists no ring R in which $5 \leq gr\,(\Gamma\,(R)) < \infty$. So, which rings have zero divisor graphs in which $gr\,(\Gamma\,(R)) = 3,\ 4$, or ∞?

- Let R be a commutative ring with 1 and U be an $R-$ module. If $diam\,(\Gamma\,(R)) > 1$, then $diam\,(\Gamma\,(R \oplus U)) > 1$. Then, which classes of rings preserve the diameter?

Recommendations

An obvious general algebraic structure on the set of zero divisors in a ring does not exist. In a commutative ring, the set of zero divisors is a multiplicative semigroup. It is for this reason that nonalgebraic methods including the zero divisor graphs have been used to study sets of zero divisors. In order to establish the relationship between a finite ring R and the graph $\Gamma(R)$, the isomorphism and classification problems arise naturally. This thesis has provided partial solutions to these two problems on the classes of rings described by Construction I and II. We recommend that further research should address the two problems on other classes of commutative finite rings. We in particular recommend that further studies be done to answer the following:

- Let R be a ring described by Construction I, what is the possible Construction S such that $\Gamma(R) \cong \Gamma(S)$?

- Let R' be a ring described by Construction II, what is the possible Construction S' such that $\Gamma(R') \cong \Gamma(S')$?

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