A COMPARATIVE STUDY OF COX-INGERSOLL-ROSS AND VASICEK MODELS: A CASE STUDY OF THE 91-DAY TREASURY BILL RATE IN KENYA

BY

MERCY JEPCHUMBA MSC/MAT/00026/021

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN ACTUARIAL SCIENCE

SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

MASENO UNIVERSITY

DECLARATION

This project is my own work and has not been presented for a degree award in any other institution.

MERCY JEPCHUMBA

MSC/MAT/00026/021

Signature:...................................Date:...............................

This project has been submitted for examination with my approval as the university supervisor.

DR. JOSHUA WERE

Department of Statistics and Actuarial Science

Maseno University

Signature:.....................................Date:..............................

ACKNOWLEDGEMENT

I would like to thank God for giving me knowledge,good health and the strength for completing my project. I want to express gratitude for my Supervisor Dr Joshua Were for his guidance, continuous support and invaluable advice.I also acknowledge my entire family for supporting and encouraging me.

Dedication

To my mum Martinah,my sister Joy and my brother Carson for their unwavering support, encouragement and belief in me, I would always be indebted to you all. I am also gratefull to the extended family and friends.

ABSTRACT

Interest rates is an important factor in the operation of any financial market, with different interest rates having different effects on investment decisions. As such, understanding how interest rates move across different markets can be a crucial factor in managing market risk and maximizing returns. The success of financial investments heavily relies on accurately predicting changes in the rates of interest. The objective of the research was to compare between the Vasicek andCIR model, more accurately captures dynamics of interest rates in Kenya. The reason for choosing these models is they are commonly used because they are analytically tractable and easy to implement. To achieve the objective of this study, we estimated parameters for the models.We compared the performance of both models in predicting future interest rate values. Data on the Treasury bill rates with 91 days maturities was used from the website of the CBK as a proxy of interest rate from July 2019 to September 2023. Parameters were derived using the Ordinary Least Squares technique. An advantage of using the method is that it is easy to implement and handles large data sets efficiently. Microsoft Excel was employed for data simulation. The estimates obtained from both the CIR and Vasicek Models were then used to determine which one better fit the available data, with the research recommending the use of the Vasicek Model due to its stable simulated data and the absence of any significant difference between its test statistics and the actual data. However, caution should be exercised when applying both the Vasicek and CIR models. These findings can serve as a foundation for developing more effective predictive tools for forecasting future interest rate values in Kenya, enhancing the accuracy and robustness of financial analysis and research in this domain.

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CHAPTER 1

INTRODUCTION

1.1 Background to the Study

Utilizing a stochastic approach for modeling interest rates is of paramount importance in the field of investments. When a reliable model for interest rate determination is established, it simplifies the process of setting aside reserves for insurance companies and managing returns for pension schemes. Moreover, it enables more accurate pricing of financial products, given that future value of money is a critical component of investments and heavily dependent on the credited interest rate. Stochastic modeling involves employing mathematical models to simulate future events and quantify the associated risks in specific investments. This approach proves particularly valuable for predicting interest rate behavior due to the highly volatile and un- predictable nature of interest rates. A stochastic model, in essence, is a mathematical framework that accounts for the inherent randomness and uncertainty of future events and employs probability theory to predict their outcomes. Brownian Motion stands as a pivotal stochastic process widely used in stochastic modeling. Originally introduced by Brown to study the random movement of pollen in liquids, it was subsequently adopted by Bachelier in 1900 to model stock prices. Ornstein- Uhlenbeck, thereafter, represents a continuous stochastic process characterized by its mean-reversion property. Therefore, it garners approval from both scholars and professionals for modeling the yield curves of interest rate variations. Consequently, this research endeavor sought to examine the utilization of stochastic models in representing fluctuations in interest rates. The primary focus of this research project revolved around single factor short rate models, which bases on the concept that interest rate fluctuations arise from a change in a solitary underlying random factor. The models in the study were selected for their convenience, as they provide readily calculable solutions, making implementation more straightforward.

1.2 Basic Concepts

A **treasury bill** is a short-term, paperless borrowing instrument that is issued to raise funds from institutional investors and the public by the government. The TB rates are issued with maturity periods of 182,91 and 364 days and thereafter sold at discounted prices to reflect the investor's returns and are redeemed at its face value(value at maturity). The differences of TB rate face value and its discounted rate represents the returns to investors.The discounted prices of these Treasury Bills depend on the rate that is quoted and calculated as below:

$$p = 100(\frac{1}{1 + (r * \frac{d}{365})}) \tag{1.1}$$

where ;p is the price per Kshs.100

r is the interest rate quoted

d represents days to TB rate maturity.

A **stochastic process** is a series of random variables X_s at time t collected in a state space J. This process is usually denoted as $(X_s : t \epsilon J)$

A **Markov process** is a form of a stochastic process characterized by three fundamental traits: a finite number of possible outcomes or states, a property where the outcome at each stage depends exclusively on the outcome observed at the previous stage, and the maintenance of constant probabilities for state transitions over time.A process X_t has a Markov property if;

$P(X_t \leq x|F_s) = P(X_t \leq x|X_s)$ for all $t > s \geq 0$

given that F_s represents filtration that is associated with that stochastic process. Filtration describes information that has been gained upto time s.

It is a process that having knowledge of current state in that process provides necessary information needed to calculate probabilities of future states.

Stochastic modelling- Stochastic modelling is a type of mathematical modelling that uses probability theory and random variables to model processes that involve random uncertainty. It involves using mathematical models to simulate future events and quantify the risk of certain investments. It is used to predict the future probability of events by analyzing past trends and data. In order to develop the theory of stochastic modelling, we will present the Mathematical explanations to stochastic models later in chapter three.

1.3 Statement of the Problem

Financial markets are constantly changing and evolving, and as a result, interest rates are subject to unexpected and unforeseen changes. As such, it is critical that stakeholders are able to accurately forecast interest rate changes to ensure that they can make informed decisions that will protect their investments and ensure the sustainability of their businesses. Stochastic models represent potent and precise instruments for predicting fluctuations in interest rates. Despite the wide variety of stochastic interest rate models available,the Vasicek and CIR models are commonly used.Also, there remains a significant need to continuously evaluate which model most effectively encapsulates the intricacies of interest rate movements in Kenya using the current data and the current market.Therefore,this project uses data from the last four years for analysis.

Over the years,several methods have been used for parameter estimation.These methods includes the Maximum Likelihood Estimation(MLE),General Method of Moments and the Ordinary Estimation method(OLS). Each method has some shortcomings and therefore this study will use the Ordinary Estimation method(OLS) which hasn't been used widely in past research in Kenya. An advantage of this method is that it is easy to implement and handles large datasets efficiently. The identification of these research gaps serves as the primary objective of this study.

1.4 Objectives of this Study

The aim of the research was assessing the effectiveness and performance of Cox-Ingersol-Ross and Vasicek equilibrium,single factor short rate models in the context of 91 days maturity treasury bill rate in Kenya. The specific goals were as follows:

1. Estimating models' parameters for the specified equilibrium models.
2. Utilize these parameters in the models to generate simulated interest rate values.
3. Analyze the performance of the two models in predicting future interest rate values.

1.5 Significance of the Study

This study's outcomes would offer valuable insights for policymakers in the Ministry of Finance , the Central Bank of Kenya and the Treasury aiding them in formulating policies related to interest rate regulations and establishing base rates for various financial instruments within the financial markets. Additionally, financial analysts working in investment banks, commercial banks, and risk management roles will benefit from this research by gaining a deeper understanding of how to monitor changes in interest rates. This knowledge will empower them to provide guidance to financial sector stakeholders on how to mitigate risks associated with interest rate fluctuations. It will also be instrumental in making investment decisions based on observed patterns of interest rate volatility and projecting returns from pension fund investments. Scholars and researchers interested in studying the volatility of financial instruments such as bonds, treasury bills, stocks, and foreign exchange rates will find the study's findings highly relevant. These findings can serve as a foundation for developing more effective predictive tools for forecasting future interest rate values in Kenya, enhancing the accuracy and robustness of financial analysis and research in this domain.

CHAPTER 2

LITERATURE REVIEW

This literature review explores related articles and can be separated into two main aspects: parameter estimation methods and the application of these methods in previous studies, as well as past research on short rate modelling with focus on single factor equilibrium models.

2.1 Parameter Estimation Methods

In their paper Chan et al. (1992) evaluate and analyze a range of continuous time short-rate models using the Generalized Method of Moments to find out which model best fits the short-term interest rate data. They use one-month Treasury bill yield data covering the period June 1964 to December 1989. They observe that the most effective models in capturing the short-rate dynamics are those that allow for high sensitivity of interest rate changes to the level of interest rate. In their study several reputable short-rate models exhibit poor results due to their restrictions on the volatility of the term structure. They demonstrate that the results of their analysis have necessary ramifications for the use of various short-rate models in hedging of interest rate risk and also in pricing of interest rate contingent claims.(Chan,2014)

Gupta and Zeytun (2007) did a Comparative Study of the Vasicek and the Cox,Ingersol,Ross(CIR) models. They used the Least-Squares technique to estimate parameters for the models. They used the actual Canadian zero-coupon bond data from January 1997 to December 2006 . Results showed that the Vasicek model seemed to perform better due to the more stable volatility parameter. The results of the two models seemed quite similar, but the calibration gave a higher sigma in the CIR model, which resulted in unstable rates.(Zeytun,2007)

Nowman (2011) uses Gaussian econometric estimation methods to evaluate the stochastic differential equation models for the interest rate dynamics of the United Kingdom bond market.His work used monthly data over the period from 1970 to 2010 utilizing a variety of maturities.The results of single and two equations models Gaussian estimates suggest that the volatility of rates depend on the level of rates across the maturities. Moreover, the study observes that there is no empirical evidence of mean reversion in the interest rates in the market. In addition,the CIR-SR and CKLS models have empirical support in the United Kingdom bond market.(Nowman,2011)

Khramov (2013) did a comparative study on estimating parameters of short-term interest rate models. He used the Generalized Method of Moments(GMM) in estimating parameters for ten short rate models. He used US 3-month Treasury-Bill interest rate from January 1978 to December 2012. The model comparison confirmed that the Cox, Ingersoll, and Ross (CIR) model provided a good characterization of the short-term interest rate process.(Khramov,2013)

Abid and Chakroun (2014) developed a method for estimating the short-rate yield curve in Tunisia's bond market. They assessed the performance of the Vasicek and CIR models in predicting interest rate dynamics, utilizing the Maximum Likelihood Estimation with ordinary least squares method. Their results indicated that estimates of the model's parameters produced upward-sloping yield curves, with the Vasicek model exhibiting superior performance in replicating short-term rates within Tunisia's bond market (Chakroun, 2014).

Zhao and Wang (2017) conducted a study comparing the performance of the Vasicek and CIR models in United States, New Zealand, and United Kingdom. Their analysis used the General Method of Moments parameter estimation method and historical data, including the Treasury bill rates with a maturity of 91 days for United Kingdom and United States, as well as the bank rates with a maturity of 30 days for New Zealand. Their conclusion was that the Cox-Ingersoll-Ross model provided a better fit for their data compared to the Vasicek model. Furthermore, both models were found to effectively capture the long-term dynamics of interest rates (Zhao, 2017).

Chelimo (2017) utilized Ordinary Least Squares technique in calibrating Vasicek model for evolution in interest rate dynamics in Kenya. In his research, Chelimo employed multi-states and single-states modeling within Hidden Markov process, using the three-months TB rates which acted as proxies for interest rates in the short term. His findings indicated that an increased number for these states resulted to increases for the value of mean-reverting parameters. Additionally, Chelimo's study suggested that the level of the interest rate did not significantly influence volatility, emphasizing the importance of incorporating regime switches in interest rate models (Chelimo, 2017).

Miao (2018) did a comparative study of Vasicek and CIR and CKLS models. He used the ordinary least squares method to estimate parameters for the models and an initial start value for implementation of a numerical estimate of parameters that maximize the likelihood. He used the data sample from PRIBOR (Prague interbank offered rate) 3 months maturity data . The empirical interest rate data were non-negative and fluctuated more for larger interest rates than for interest rates close to zero. That behavior was captured by the CIR model but not by the Vasicek model. Hence the CIR model seemed to be a more appropriate interest rate model than the Vasicek model. It was also observed that a CKLS model might fit empirical interest rates even more, but then it might be harder to estimate the parameters due to the lack of explicit transition densities. (Miao,2018)

Orlando, Bufalo and Mininni (2019) did a study on forecasting interest rates using Vasicek and CIR using a partitioning approach. They used the Maximum Likelihood Estimation Method to estimate parameters. They used weekly dataset spanning from 31 December 2010 to 18 November 2016 EUR interest rates. The error analysis highlighted a better performance of the proposed procedure with respect to CIR Model.(Mininni,2021)

In his research, Maina (2021) conducted a comparative study of interest rate models using data from January 2005 to July 2016. Maina used TB rates having 91-day maturities as proxy for interest rates and employed Generalized Method of Moment parameter estimation method. His findings revealed weak evidences of the mean reverting feature of the models he had chosen. Notably, Maina's study established a positive relationship between short rates level and short rates volatilities. Chan, Karolyi, Longstaff, Sanders(CKLS) model emerged as the best-performing model, primarily due to its ability to capture the high dependency of short rates volatility on interest rates level. Moreover, the CKLS model exhibited superior forecasting of volatility capabilities compared to the other models under study (Maina, 2021).

Ramaroson (2022) did a study on Stochastic Interest Rate Models for forecasting the term structure of interest rates of Kenya using the Maximum Likelihood Estimation Method (MLE). He used weekly Kenya Bond Yield starting from 14 April 2012 until 12 May 2022 .He did calibration of Vasicek and Cox–Ingersoll–Ross models by using the partitioning approach. The CIR model fit well the data better than the Vasicek by applying the proposed numerical procedure on the data-sets.(Ramaroson,2022)

2.2 Term structure of interest rates

Term structure defines the relationship of rates of interests and yields of bonds having different maturity periods.

The approaches for models that are used in defining short rates include:

- The Heath-Jarow-Mortton(HJM) assumes the Ito processes for modelling forward rates in investments having constant maturities.
- A Term structure model assumes the Ito processes for modelling short rates. Examples of such models include Cox-Ingersol-Ross models, Vasicek model and the Hul-White models.

Therefore short rates r_s represents the rates of interest credited for the shortest periods denoted by;

$r_s = r\left(s, s+\delta\right) \approx\ R\left(s, s+\delta\right)$

and δ represents the smallest positive number.

Desirable features of term structure models.

When selecting an interest rate model, several desirable characteristics should be considered:

- Arbitrage-Free: That model must be arbitrage-free, meaning it must not allow for riskless profit opportunities through dynamic hedging. In practice, bills/ bonds and derivatives of short rates have always been assumed that they are arbitrage-free.
- Positive Rates: These models should produce positive rates of interest. Banks should be able to offer investors positive returns to discourage hoarding cash. While this may be impractical for certain financial institutions like pension funds and large life offices, it

holds in practice. Some models, such as the Vasicek model, do allow for negative interest rates.

- Mean-Reverting Behavior: The model is assumed to exhibit certain degree of the mean-reversion behavior.Previous studies shows that short rate models are mean-reverting in practice. That may not feature as a highly mean reverting behavior,however, the assumption is necessary in acturial applications.

- Computational Feasibility: This means the ease to evaluate the price of bills and other derivatives using those models. It's crucial that the model allows for efficient hedging calculations and pricing, and it should enable the identification of potential arbitrage opportunities and quick rebalancing of hedged positions. Therefore, models that facilitate price computation using numerical techniques or straightforward formulae for option and bond prices are preferred.

- Realistic Dynamics: The model should produce realistic dynamics, capturing features similar to those observed in historical data with a reasonable probability. It should also be capable of generating a range of realistic yield curves, including upward sloping, humped, or downward sloping curves.

- Historical Data Fit: The model should fit historical data of interest rates, with parameters appropriately estimated to match past market behavior.

- Calibrating into market's values: This model should be easily calibrated to prevailing market data. Calibration should not just provide a good approximation but should be precise in establishing a fair value for liabilities. When the model doesn't adequately fit observed yield curves, it cannot reliably determine the fair value of liabilities.

- Flexibility: The model should be flexible and capable of accommodating a variety of derivatives and financial instruments.

- Suitability for Monte-Carlo Simulations: Those models should easily suite well for Monte-Carlo simulations, a widely used technique for pricing and risk management in finance

2.2.1 Review of term structure models

Most common approaches in modelling short rates is assuming that the short rate follows continuos-period Markov processes. In 1973,Merton proposed an interest rate model. The model belonged to Gausian group of modelling stochastic processes that follows Arithmetic Brownian motion. It allowed negative values for the short rate models and assumes a constant risk premium.

The Vasicek (1977) model made a substantial impact on modelling of interest rate values. Vasicek proposed a single factor equilibrium model with several assumptions. These assumptions are that the short rate followed diffusion processes, that prices for bonds depend on interests rates and that they happen in efficient markets. The model depends on arbitrage theory and allows for negative interest rate values(Vasicek ,1977)

Dothan (1978) introduced a model that explains the relationship existing in the levels of the short rate to the volatilities of the short rate more strongly. Term structure under this model results in a decreased factor for the period upto maturity and increased factors for the interests rate. Brennan in collaboration with Schwartz in 1980 made an extension of the Dothan model in introducing mean reversion parameter. Therefore, distribution of that interest rate is not known with the price of a contingent claims computed using numerical methods.(Dothan,1978)

Cox et al. (1985) introduced the CIR model that extends Vasicec model with a new parameter introduced,that is square-root of variance parameter. This model follows squareroot mean reverting process. This feature makes it ideal for modelling interest rates because it does not allow for negative values.(Cox,1985)

Ho-Lee (1986) introduced a term structure model which assumes arbitrage opportunities. Short-rate for the model relates the instantaneous rate by a random variable and defined as binomial trees.(Ho,1986)

Hull-White in 1990 made an extension of Cox-Ingersol-Ross model and Vasicec model in introducing a time-dependent drift which culminated to models which are consistent to the price of bonds in the market.(Hull,1990)

Black et al. (1990) proposed a model which takes yield volatilities and existing structure for the zero-coupon yield as inputs for bonds using the binomial lattice framework.(Black ,1990)

Name of short rate	Stochastic Diferential Equations
Vasicek	$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma\ dW_t$
Merton	$dr_t = \mu dt + \sigma\ dW_t$
CIR	$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma\sqrt{r_t}dW_t$
Hull-White	$dr_t = \alpha\left(\mu\left(t\right) - r_t\right)dt + \sigma\ dW_t$
Hull-White-Cox-Ingersol-Ross	$dr_t = \alpha\left(\mu\left(t\right) - r_t\right)dt + \sigma\ dW_t$
Dothan	$dr_t = \mu r\left(t\right)dt + \sigma\ r_t dW_t$
Ho-lee	$dr_t = \mu_t dt + \sigma\ dW_t$
Black-Derman-To-Model	$dr_t = \alpha\left(t\right)\left(\mu - r_t\right)dt + \sigma_t dW_t$
Black-Karansinski	$dr_t = r_t\alpha_t - \beta_t ln_t dt + \sigma_t r_t dW_t$

CHAPTER 3

Research Methodology

The section explains the Mathematical background to stochastic models,parameter estimation method for the models under study using Ordinary Least Squares (OLS) technique and simulation of interest rates using the discrete versions of the model's Stochastic Differential Equations.

3.1 Mathematical background to Stochastic Models

3.1.1 Weiner process

In 1828, the botanist Robert Brown observed that pollen particles suspended inside liquids displayed irregular and random motions. Building on this, Albert Einstein in 1905 suggested that this erratic movement could be described using equations associated with Brownian motion. Later, in the year 1900, Louis Bachellier applied Brownian Motion to model the fluctuations in stock prices.

The formal Mathematical framework for Brownian motion modelled as stochastic processes had been established by Norbert W'iener in 1923. These stochastic processes are occasionally referred to as the Wiener process in his honor.

Stochastic processes ,say W are considered Wiener processes under the conditions given below:

- $\mathrm{W}(0) = 0$ that is, value at time 0 is 0

- That process has independent increments, that is, for $r < s \leq t < u$ therefore $W(u) - W(t)$ and also $W(s) - W(r)$ will therefore be both independent stochastic variables.
- This stochastic variable $W(t) - W(s)$ is a Gausian distribution ,that is, $N\left(0, \sqrt{t-s}\right)$
- The process results in continuous trajectories.

3.1.2 Ito's Lemma

When striving to develop calculus used for processes like Brownian motion , a significant challenge arises due to the non-differentiable nature of their sample paths. Attempting a straightforward approach to stochastic integrals proves to be unfeasible. However, Kiyoshi Ito, who conducted his work independently in Japan, made a pivotal breakthrough. He recognized that functions of Wiener processes would be twice differentiable and measurable concerning natural filtration of a Wiener process. Furthermore, by taking advantage of its property that independent increment is exhibited by Wiener processes, he introduced Ito's Lemma as a valuable mathematical tool.The Lemma states that when X has a stochastic process given by:

$dX_t = \mu dt + \sigma dW_t$ given that;

μ represents model's local drift parameter, σ represents model's diffusion parameter given that μ ,σ are adapted processes then the function $f(t, X_t)$ has the differential equations represented as:

$$df\left(t, X_t\right) = \frac{df}{dt} + \mu\frac{df}{dX_t} + \frac{1}{2}\sigma^2\frac{d^2 f}{dX_t{}^2}dt + \sigma\frac{df}{dX_t}dW_t \tag{3.1}$$

3.1.3 Stochastic Differential Equation (SDE)

Equations given by:

$$dX_t = \mu dt + \sigma dW_t \tag{3.2}$$

whereby functions σ and μ are given and X_t is a stochastic process ,then dX_t is said to be a Stochastic Differential Equation driven by a Weiner process. The functions σ and μ represents the drift and diffusion coefficients respectively.

3.1.4 Geometric Brownian Motion (GBM)

GBM is a crucial process in finance for modeling stock prices, and it resembles the Ornstein-Uhlenbeck process. In this continuous-time stochastic framework, logarithm of that variable under consideration follows Brownian motion.

Consider the function

$$f(t, X_t) = \ln(X_t) \tag{3.3}$$

which follows a GBM with the dynamics

$$dX_t = \alpha\, X_t dt + \sigma\, X_t dW_t \tag{3.4}$$

whereby α and σ are positive constants. To solve this SDE take the following steps:

• Take the partial derivatives of the function in (3.3)

$\frac{df}{dt} = o$, $\frac{df}{dX_t} = \frac{1}{X_t}$, $\frac{d^2 f}{d{X_t}^2} = -\frac{1}{{X_t}^2}$

Substitute the values above to Ito's formula below:

$$df(t, X_t) = \frac{df}{dt}dt + \frac{df}{dX_t}dX_t + \frac{1}{2}\frac{d^2 f}{d{X_t}^2}(dX_t)^2 \tag{3.5}$$

$df(t, X_t) = \frac{1}{x}dX_t - \frac{1}{2x^2}(dX_t)^2$

Substitute the value of dX_t with equation (3.4) above

$$df\left(t, X_t\right) = (\frac{1}{X_t}\left(\alpha X_t dt\right) + \left(\sigma X_t dW_t\right) - \frac{1}{2X_t^2}\alpha^2 X_t^2\ \left(dt\right)^2) + (\sigma^2\left(X_t\right)^2\left(dW_t\right)^2) + (2\alpha\sigma\left(X_t\right)^2 dW_t dt\ ) \tag{3.6}$$

Apply the assumptions below under Ito's Lemma to equation (3.6) above

$d_t * d_t = 0$

$dt * dW_t = 0$

$dW_t * dW_t = dt$

$df\left(t, X_t\right) = \left[\alpha - \frac{1}{2}\sigma^2\right] dt + \sigma dW_t$

Integrate both sides of the equation

$$f\left(X_t, T\right) - f\left(X_0, 0\right) = (\alpha - \frac{1}{2}\sigma^2)\int_0^t ds + \sigma\int_0^t dW_s \tag{3.7}$$

But $f\left(t, X_t\right) = \ln(X_t)$

Substituting that to equation (3.7) ,taking the exponential on both sides of the equation and making X_t the subject then the solution to the above GBM is given by

$$X_t = X_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t} + e^{\sigma W_t} \tag{3.8}$$

3.1.5 The Ornstein Uhlenbeck Process

This Ornstein Uhlenbeck process that was named from George Eugen Uhlenbeck and Leonard Ornstein, was originally developed to depict the velocities of heavy Brownian motion particles subjected to friction. This can be seen as a modification of a Wiener process, possessing properties that tend to bring it back to equilibrium.

In the context of finance, this process suggests that observed phenomena like asset prices and the volatility of the returns have tendencies of reverting to its future-term mean levels with time. The Ornstein-Uhlenbeck (OU) process is commonly employed for modeling stochastic interest rates,commodity prices and exchange rates.An O'rnstein Uhlenbec process is given by a Stochastic differential Equation:

$$dX_t = -\gamma\, X_t dt + \sigma\, dW_t \tag{3.9}$$

In finding a solution of that stochastic differential, consider a function

$$f\,(t, X_t) = X_t e^{\gamma t} \tag{3.10}$$

Find the partial derivatives of equation (3.10)

$\frac{df}{dt} = \gamma\, X_t e^{\gamma t}$, $\frac{df}{dX_t} = e^{\gamma t}$, $\frac{d^2 f}{d{X_t}^2} = 0$

Substitute the values to Ito's formula equation (3.5)

$df\,(t, X_t) = \gamma\, X_t e^{\gamma t} dt - \gamma\, X_t e^{\gamma t} dt + \sigma\, e^{\gamma t} dW_t$

$df\,(X_t, t) = \sigma\, e^{\gamma t} dW_t$

Integrating the two sides of the equation between times 0 to T:

$$f\,(X_t, T) - f\,(X_0, 0) = \int_0^T \sigma e^{\gamma t} dW_t \tag{3.11}$$

But $f\,(X_t, t) = X_t e^{\gamma t}$

Substituting this to equation (3.11) the equation becomes:

$X_T e^{\gamma T} = X_0 + \sigma \int_0^T e^{\gamma t} dW_t$

Dividing through by $e^{\gamma T}$ to make X_T the subject ,then the solution to this Stochastic Differential Equation is:

$$X_T = X_0 e^{-\gamma T} + \sigma \int_0^T e^{-\gamma(T-t)} dW_s \tag{3.12}$$

with its variance and mean given by:

$E\left[X_T\right] = X_0 e^{-\gamma T}$

$Var\left[X_T\right] = \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma T}\right)$

3.1.6 The Mean-reverting process

This is a generalization of Ornstein Uhlenbec process. When this process moves away from the long-run mean,a component pulls it back towards its mean at velocity estimated from $\gamma > 0$. The process is used for modelling the short rate which is assumed to be a mean-reverting process.

This process is defined by a stochastic differential equation:

$$dY_t = \gamma\left(\mu - Y_t\right) dt + \sigma\, dW_t \tag{3.13}$$

which bases on O'rnstein Uhlenbec(OU) processes.

To solve this SDE, consider the function

$$df\left(Y_t, t\right) = e^{\gamma t} Y_t \tag{3.14}$$

First find the partial derivative of the function in (3.14)

$\frac{df}{dt} = \gamma\, Y_t e^{\gamma t}$, $\frac{df}{dY_t} = e^{\gamma t}$, $\frac{d^2 f}{dY_t{}^2} = 0$

Substitute the values into Ito's formula (3.5) and substituting for the value of dY_t in (3.13) ;

$df\left(t, X_t\right) = \mu\gamma e^{\gamma(t)} dt + \sigma\, e^{\gamma t} dW_t$

Integrating the two sides of the equation between times 0 to T:

$$f(X_t, T) - f(X_0, 0) = \mu\gamma \int_0^T e^{\gamma s} ds + \sigma \int_0^T e^{-\gamma s} dWs \tag{3.15}$$

But $f(Y_t, t) = e^{\gamma t} Y_t$

Substituting this into the equation(3.15) becomes:

$e^{\gamma t} Y_T = Y_0 + \mu(e^{-\gamma t} - 1) + \sigma \int_0^T e^{-\gamma(s)} dW_s$

Dividing through by $e^{\gamma t}$ to make Y_T the subject,then a solution to this mean reversion process above is :

$$Y_T = Y_0 e^{-\gamma t} + \mu\left(1 - e^{-\gamma t}\right) + \sigma \int_0^T e^{-\gamma(s)} dW_s \tag{3.16}$$

The variance and expectation of the process is given by:

$$E[Y_T] = Y_0 e^{-\gamma t} + \mu\left(1 - e^{-\gamma t}\right) \tag{3.17}$$

$$Var(Y_T) = \frac{\sigma^2}{2\gamma}\left(1 - e^{-\gamma(T-t)}\right) \tag{3.18}$$

3.1.7 Square root mean reversion process

The process looks similar to O'rnstein Uhlenbec mean reversion process. However, given parameters $\sigma \leq \gamma\mu$ this proces becomes positive.When this O'rnstein-Uhlenbec process reaches 0, it deterministically moves away from it. This characteristic is highly beneficial for modeling interest rates and asset prices that are expected to remain positive. This process is defined from its Stochastic Differential Equation;

$$dX_t = k(\mu - x_t) dt + \sigma\sqrt{X_t} dW_t \tag{3.19}$$

To find the solution of this feature, consider a function

$$f\left(t, X_t\right) = e^{kt} X_t \tag{3.20}$$

Find partial derivative of the function in (3.20) and apply Ito's formula from (3.5)

$\frac{df}{dt} = ke^{kt} X_t$, $\frac{df}{dX_t} = e^{kt}$, $\frac{d^2 f}{dX_t{}^2} = 0$

Substituting for the value of dX_t from equation (3.19)

$df\left(t, X_t\right) = ke^{kt} X_t dt - ke^{kt} X_t dt + k\mu e^{kt} dt + \sigma e^{kt} \sqrt{X_t} dW_t$

$df\left(t, X_t\right) = k\mu e^{kt} dt + \sigma e^{kt} \sqrt{X_t} dW_t$

$f\left(X_t, T\right) - f\left(X_0, 0\right) = \int_0^T k\mu e^{kt} dt + \int_0^T \sigma e^{kt} \sqrt{X_t} dW_t$

There is no solution for X_T in closed form. However, the conditional mean and variance are as shown below :

$$E\left[X\left(t\right)\middle|\ F\ s\right] = X\left(s\right) e^{-k(t-s)} + k\left(1 - e^{-k(t-s)}\right) \tag{3.21}$$

$$Var\left[X\left(s\right)|Fs\right] = X\left(s\right) \frac{\sigma^2}{k} \left(e^{-k(t-s)}\right) - e^{-2k(t-s)} + \mu \frac{\sigma^2}{2k} \left(1 - e^{-k(t-s)}\right)^2 \tag{3.22}$$

3.2 Model Specifications

3.2.1 The Vasicek Model (1977)

Vasicek introduced a one-factor model that posits that only market risk influences the movement of interest rates. The Vasicek model assumed that the short rate follows an Ornstein-Uhlenbeck process, effectively capturing mean-reverting nature of interest rates.

In practical terms, that means that if the current rates deviate from the future mean $r > \mu$, then the parameter coefficient α induces a negative drift, causing the rate to move downward towards mean .Alternatively, if short rate is below its long-term average $r_t < \mu$, α imparts a positive drift, causing the rate to gravitate upward toward μ. The coefficient represents the speed for which interest rates adjust to reach the long-term mean level.

Economic rationale supports this notion of the mean reverting behavior. If interest rate is high, economic activity will tend to be slow , leading to decreased borrowing. As a result, interest rates are pulled back towards an equilibrium level, causing them to decline. Conversely, when interest rates are low, there is high demand for funds from borrowers, which tends to push rates higher.

Another notable feature of the Vasicek model is the tractability of the model and also the availability of closed form solutions even for more complex financial interest rate derivatives.

A major disadvantage of Vasicek model is the possibility of negative interest rates which can yield some illogical results and prices.

Vasicek's Stochastic Differential Equation is given by:

$$dr_t = \alpha\left(\mu - r_t\right) dt + \sigma_t \, dW_t \tag{3.23}$$

In finding a solution to the Stochastic differential equation, consider the function

$$f\left(t, X_t\right) = r_t e^{\alpha t} \tag{3.24}$$

find the partial derivatives of equation (3.24)

$\frac{\partial f}{\partial r} = e^{\alpha t}$, $\frac{\partial f}{\partial t} = r_t \alpha \, e^{\alpha t}$, $\frac{d^2 f}{dr^2} = 0$

Using Ito's formula below:

$$df\left(t,r_t\right)=\frac{df}{dt}dt+\frac{df}{dr}dr_t+\frac{d^2f}{dr^2}\left(dr_t\right)^2 \tag{3.25}$$

Substitute the partial derivatives to Ito's formula in (3.25) $df\left(t,r_t\right)=\alpha\ r_te^{\alpha t}dt+\mu\alpha\ e^{\alpha t}-\alpha\ r_te^{\alpha t}dt+\sigma\ e^{\alpha t}dW_t$

$$df\left(t,r_t\right)=\mu\alpha\ e^{\alpha t}+\sigma\ e^{\alpha t}dW_t \tag{3.26}$$

Integrate both sides of equation(3.26)

$$f\left(X_t,T\right)-f\left(X_0,0\right)=\mu\alpha\int_0^t e^{s\alpha}ds+\sigma\int_0^t e^{s\alpha}dW_s \tag{3.27}$$

But $f\left(t,X_t\right)=r_te^{\alpha t}$

Substituting into the equation (3.27) becomes:

$e^{\alpha t}r_T=r_0+\mu\left(e^{-\alpha t}-1\right)+\sigma\int_0^T e^{-\alpha(s)}dW_s$

Dividing through by $e^{\alpha t}$ to make r_T the subject ,then a solution to this mean reversion process above is :

$$r_T=r_0e^{-\alpha t}+\mu\left(1-e^{-\alpha t}\right)+\sigma\int_0^T e^{-\alpha(t-s)}dW_s \tag{3.28}$$

Taking the expectation on both sides of equation (3.23) :

$$E[r_T]=E[r_0e^{-\alpha t}]+E[\mu\left(1-e^{-\alpha t}\right)]+E[\sigma\ e^{-\alpha t}e^{\alpha s}W_t] \tag{3.29}$$

Expectation of a Weiner process in the last part becomes zero.Therefore:

$$E\left[r_T\right]=r_0e^{-\alpha t}+\mu\left(1-e^{-\alpha t}\right) \tag{3.30}$$

Taking the variance of equation (3.23) gives the variance of r_t as:

$$Var\left[r_T\right] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \tag{3.31}$$

Where r_0 represents the interest rates value during the starting period and r_T represents interest rates value for future time t.

3.2.2 Cox-Ingersoll-Ross Interest rate model (1985)

The model was proposed by Cox,Ingersoll and Ross which was an extension of Vasicek interest rate model. It is a form of a single factor equilibrium model, that means, it explains randomness of short rates as driven by a single factor. CIR is used for valuing the derivatives of interest rates. This model assumes mean reverting behaviour of interest rates towards future average levels of interest rate values.

The CIR model uses the square-root mean reverting process. This ensures interest rates remain positive which is the reason why some authors prefer to use it in pricing derivatives of interest rates.

A disadvantage of CIR model is that it has no closed form solution for r_t which makes it hard to value complex financial interest rate derivatives.

The Stochastic Differential Equation of the model is given by:

$$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma\sqrt{r_t}dW_t \tag{3.32}$$

To find the solution of the Stochastic Differential Equation , consider a function

$$f\left(t, r_t\right) = e^{\alpha_t} r_t \tag{3.33}$$

Find partial derivative from this function in (3.33) and apply Ito's formula from (3.25)

$\frac{df}{dt} = \alpha e^{\alpha_t} r_t$, $\frac{df}{dr_t} = e^{\alpha t}$, $\frac{d^2 f}{d{r_t}^2} = 0$

Substituting for the value of dr_t from equation (3.32)

$df\left(t, r_t\right) = \alpha e^{\alpha t} X_t dt - \alpha e^{\alpha t} X_t dt + \alpha \mu e^{\alpha t} dt + \sigma e^{\alpha t} \sqrt{X_t} dW_t$

$$df\left(t, r_t\right) = \alpha \mu e^{\alpha t} dt + \sigma e^{\alpha t} \sqrt{X_t} dW_t \tag{3.34}$$

Integrating both sides of equation (3.34) :

$f\left(X_t, T\right) - f\left(X_0, 0\right) = \int_0^T \alpha \mu e^{\alpha t} dt + \int_0^T \sigma e^{\alpha t} \sqrt{X_t} dW_t$

There is no solution for r_t in closed form. However, the distribution of CIR on the rate r_t is chi-square with conditional mean and variance given by:

$$E\left[r\left(t\right)\right|F\ s] = r\left(s\right) e^{-\alpha(t-s)} + \mu\left(1 - e^{-\alpha(t-s)}\right) \tag{3.35}$$

$$Var\left[r\left(s\right)|Fs\right] = r\left(s\right) \frac{\sigma^2}{\alpha}\left(e^{-\alpha(t-s)}\right) - e^{-2\alpha(t-s)} + \mu\frac{\sigma^2}{2\alpha}\left(1 - e^{-\alpha(t-s)}\right)^2 \tag{3.36}$$

Where r_s is the rate at initial period and r_t represents the rate at forecasted time t.

3.3 Parameter Estimation

Ordinary least squares method was used to estimate parameters for the models under study. The method aims to minimize the sum of squares of errors between the actual data and the predicted data. An advantage of the least squares method is that it is easy to implement and handles large datasets efficiently.

The least squares regression model is represented as:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + \varepsilon \tag{3.37}$$

where $\hat{\beta}_1$ represents the slope

, $\hat{\beta}_0$ represents the intercept,

ε_t represents an error term given that x,y represent two random variables.

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{3.38}$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \tag{3.39}$$

where $\bar{y}$ represents average value of y, $\bar{x}$ represents average value of x and S_{xy} ,S_{xx} given by

$$S_{xx} = \sum_{t_i=1}^{n} (x_i - \bar{x})^2 \tag{3.40}$$

$$S_{xy} = \sum_{t_i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \tag{3.41}$$

$$S_{yy} = \sum_{t_i=1}^{n} (y_i - \bar{y})^2 \tag{3.42}$$

In estimating parameters for models chosen for this study, a discrete version would be needed.

The discrete version of the Vasicek model is:

$$\Delta r_t = \alpha(\mu - r_t)\Delta t + \sigma \Delta W_t \tag{3.43}$$

The discrete version of the CIR model is:

$$\Delta r_t = \alpha(\mu - r_t)\Delta t + \sqrt{r_t}\sigma\Delta W_t \tag{3.44}$$

where r_t represents rate of interest at time t, and W_t represents a Weiner process.

The discrete versions can be solved as Ordinary Linear Regression(OLS) models.

The discrete version of Vasicek model can be written in OLS form as:

$$r_t - r_{t-1} = \alpha\mu - \alpha r_{t-1} + \varepsilon_t(0, \sigma^2) \tag{3.45}$$

The discrete version of CIR model can be written in OLS form as:

$$\frac{[r_t - r_{t-1}]}{\sqrt{r_{t-1}}} = \frac{\alpha\mu}{\sqrt{r_{t-1}}} - \alpha\sqrt{r_{t-1}} + \varepsilon_t(0, \sigma^2) \tag{3.46}$$

The estimates for OLS under Vasicek model will then be evaluated as:

$$S_{xx} = \sum_{t=2}^{n}(r_{t-1} - (\frac{1}{n}\sum_{t=2}^{n} r_{t-1}))^2 \tag{3.47}$$

$$S_{yy} = \sum_{t=2}^{n}((r_t - r_{t-1}) - (\frac{1}{n}\sum_{t=2}^{n}(r_t - r_{t-1})))^2 \tag{3.48}$$

$$S_{xy} = \sum_{t=2}^{n}((r_{t-1} - (\frac{1}{n}\sum_{t=2}^{n} r_{t-1}))((r_t - r_{t-1}) - (\frac{1}{n}\sum_{t=2}^{n}(r_t - r_{t-1})))) \tag{3.49}$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \tag{3.50}$$

$$\hat{\beta}_0 = \frac{1}{n}\sum_{t=2}^{n}(r_t - r_{t-1}) - \hat{\beta}_1(\frac{1}{n}\sum_{i=2}^{n}(r_{t-1})) \tag{3.51}$$

The estimates for OLS under CIR model will then be evaluated as:

$$S_{yy} = \sum_{t=2}^{n} (\frac{r_t - r_{t-1}}{\sqrt{r_{t-1}}} - (\frac{1}{n}\sum_{t=2}^{n} \frac{r_t - r_{t-1}}{\sqrt{r_{t-1}}}))^2 \tag{3.52}$$

$$S_{xx} = \sum_{t=2}^{n} (\sqrt{r_{t-1}} - (\frac{1}{n}\sum_{t=2}^{n} \sqrt{r_{t-1}}))^2 \tag{3.53}$$

$$S_{xy} = \sum_{t=2}^{n} ((\sqrt{r_{t-1}} - (\frac{1}{n}\sum_{t=2}^{n} \sqrt{r_{t-1}}))(\frac{r_t - r_{t-1}}{\sqrt{r_{t-1}}} - (\frac{1}{n}\sum_{t=2}^{n} \frac{r_t - r_{t-1}}{\sqrt{r_{t-1}}}))) \tag{3.54}$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \tag{3.55}$$

$$\hat{\beta}_0 = \frac{1}{n}(\sum_{t=2}^{n} \frac{r_t - r_{t-1}}{\sqrt{r_{t-1}}}) - \hat{\beta}_1(\frac{1}{n}\sum_{t=2}^{n} \sqrt{r_{t-1}}) \tag{3.56}$$

From the above equations, estimators for the three key parameters, namely μ (mean reversion rate), α (mean reversion level), and σ (volatility), based on a sample of n observed market rates are then estimated as follows:

From the equations above,parameters for the models will be estimated as follows:

$$\alpha = -\hat{\beta}_1 \tag{3.57}$$

$$\mu = \frac{\hat{\beta}_0}{\alpha} \tag{3.58}$$

$$\sigma^2 = \frac{1}{n-2}(S_{yy} - \frac{S^2{}_{xy}}{S_{xx}}) \tag{3.59}$$

3.4 Simulation

After parameters have been estimated using the market data, the estimates are used to simulate future expected interest rates. In practice,most financial data is in discrete form and therefore we used the dicrete form of the models to simulate future interest rates. Therefore, the formula for calculating future rates for Vasicek model is:

$$r_{t+1} = r_t + \alpha(\mu - r_t)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t \tag{3.60}$$

The formula for calculating future rates for CIR model is:

$$r_{t+1} = r_t + \alpha(\mu - r_t)\Delta t + \sigma\sqrt{(\Delta t)(r_t)}\varepsilon_t \tag{3.61}$$

where r_t represents the rate of interest at time n

Δt is the change in time

ε_t represents a normal random variable N(0,1) at time t

μ is the long term mean that future interest rate values revolve around

α is the speed at which interest rate values regroup around the mean

σ is the volatility of the model

CHAPTER 4

Results and Discussion

4.1 Introduction

Primary aim for the chapter is to both present and discuss results for this research. The chapter is organized to five main sections:

For the first part,this study discusses the description of data used to estimate parameters.It talks about the source of the data and the period for which data was analyzed.

Second section , presents the results for estimates of parameters, detailing the values obtained from the chosen models for the parameters.

The third section presents results of simulated interest rates, offering insights into how the models perform in generating interest rate values.

The fourth section provides summary statistics of the simulated results, giving a concise overview of the key characteristics and trends observed in the simulated data.

The last section presents a test of the accuracy of the models, evaluating how well they align with real-world data and assessing their performance in replicating interest rate dynamics.

These sections collectively provide a comprehensive view of the study's findings and contribute to an understanding of performance together with suitability of the short rates models under examination.

4.2 Data Description

In this study, the 91-day TB rate was employed as representation of interest rates for estimating parameters for the chosen single factor models.

Chapman demonstrated that the errors in estimation of parameters that resulted from use of Treasury Bill rate with 91 days maturity data as representation for interest rates because they were unobservable in the market are economically insignificant.

Data on interest rates was acquired from the website of the Central Bank of Kenya. This data was used to analyze the effectiveness of chosen models in simulating future interest rate values.This study's sample dataset consisted of 215 weekly observations, spanning from July 2019 to September 2023.

4.3 Parameter estimation

The formulas in Chapter Three were used in parameter estimation.

Table one below shows results for the parameter estimates

Table 1

•	α	μ	σ
Vasicek	0.185	8.172	1.644
CIR	0.175	8.0	5.792

We can convert the estimated parameters into economically meaningful interpretations.μ represents long run average,α represents velocity that interest rates revert to its average,$\frac{1}{\alpha}$ represents the time it takes for interest rates to revert to its long run averages and σ represents volatility.

The table above shows that Vasicek has a long-term average,$\mu = 8.172$, velocity of reverting to long term average, $\alpha = 0.185$, volatility parameter, $\sigma = 1.644$. Therefore, the model has a period of 5 weeks to revert back to its long term average value.

CIR has a long term average, $\mu = 8.0$, velocity of reverting to long term average, $\alpha = 0.175$, volatility, $\sigma = 5.792$. CIR takes approximately 7 weeks to revert to its long run average value.

From these results, the estimates for α and μ did not vary by a wide range. However, the estimated value of σ for the CIR model was higher as compared to the estimated value for the Vasicek model. This resulted in rates that were highly volatile for the CIR model.

4.4 Simulation

In order to simulate future rates,we first set the initial rate at time one as the rate at time one of our actual data (CBK rates).Therefore ,the interest rate at time n will be the value in time one of CBK data and interest rate in time $n + 1$ will be the rate in time 2 which we are evaluating. Model's parameters μ,α,σ and Δn are as explained in Chapter three and their values will remain constant.Therefore, to calculate the rates at time three using the equation (3.54) and (3.55) ,value at time $n + 1$ is the value we are evaluating and the value at time n will be the value simulated in time two.The other parameters remain constant.However, for ε_t , a dataset of 214 random numbers N(0,1) was generated which is the total number of our weekly observations minus one .The reason why we subtract one is because at time 1,we set an initial rate to use and therefore a random number was not needed.Therefore,at each time the rate is calculated,a different random number is used.This process will continue to the the last rate being evaluated.

The graph below shows results for the Treasury rate plotted in comparison to simulated values for the models. The rates are plotted against the period that the rates were observed.

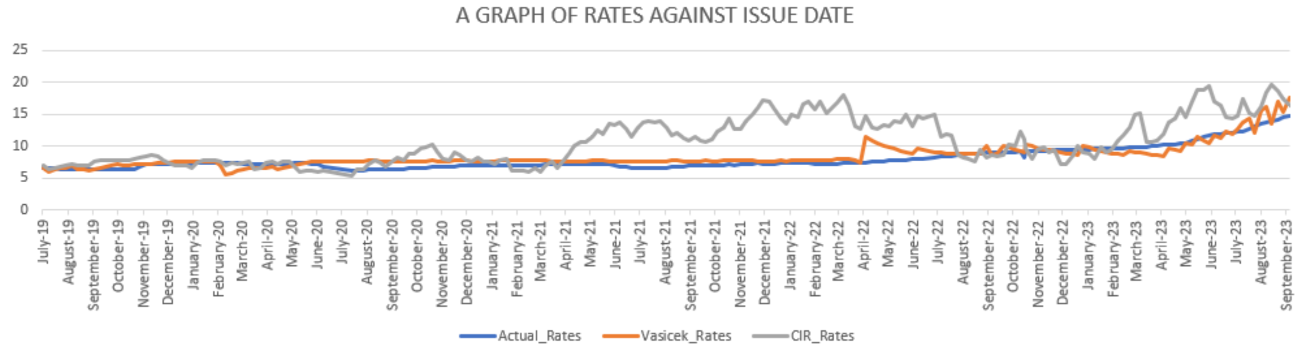

Figure 4.1: GRAPH

The plotted line of the TB rates is low from the period 2020 which was caused by Covid 19. The rates were quite stable from that period to 2022 when the rates began to rise at a high speed up to 2023.

From the plotted graph, the Vasicek seemed to have almost a similar plot to that of the actual TB rates. The plotted line for the CIR model was highly volatile due to σ having a high value.

4.5 Data Analysis

To evaluate relative performance of our models under study, the summary statistics was first evaluated.Table two provides summary statistics of the simulated and actual rates.

Table 2

•	Mean	Median	Max	Min
Actual Data	8.124	7.57	14.787	6.121
Vasicek	8.432	7.715	17.578	5.532
CIR	10.653	9.567	19.654	5.243

From this table, simulated values from the CIR model has the highest mean, that is,10.653. The mean of simulated values from the Vasicek model is 8.432 which is almost similar to the mean of our actual data of 8.124.

The results also shows that most of the test statistics of the Vasicek model are almost similar to the test statistics of our actual data.

4.6 Test of Accuracy

To compare the performances of those models in this research further, a check on the accuracy using the square-root of the Mean Square Error was used. RMSE evaluates significant changes between a statistical model's actual values and predicted values. In our case the fitted values are the simulated rates while actual rates are those rates obtained from CBK website.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_i - rf_i)^2} \tag{4.1}$$

Wherer_i denotes the actual rates and rf_i denotes corresponding fitted values.

The RMSE for the Vasicek model is 0.995160929 while that of the CIR model is 1.325.

The fact that the values of the RMSE for the Vasicek model is lower allows us to state that the model is relatively good at predicting the future interest rates. The reason for high values of RMSE is because we used one factor models which assumes that the rate of interest is affected by one random factor which is not always the case in reality. By comparing the values of RMSE of Vasicek and CIR and also from the summary statistics above, it can be affirmed that the Vasicek model is more efficient than the CIR in terms of forecasting the future movement of the Kenyan rates.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

Interest rate models play pivotal roles in risk management and investment optimization. The research focused on two widely used single-factor models, that is, CIR and Vasicek models, to assess their suitability in modeling interest rate dynamics. The objectives of this study were:Estimating model's parameters for the specified equilibrium models,utilize these parameters in the models to generate simulated interest rate values and analyze the performance of the two models in predicting future interest rate values.The parameters for the models were estimated using the Ordinary Least Squares Method.The discrete versions of the models were then used to simulate future interest rates.The models were then compared to analyze their perfomances using the summary statistics of the simulated data and also the Root Mean Square Error compared for the two models (RMSE).The estimation of model parameters revealed that while both models shared similar values for the long term average value μ and velocity of reverting to its average value α ,Cox Ingersol Ross exhibited higher values for σ , resulting in more volatile interest rates. However, high value of σ for Vasicec would potentially lead to negative rates, an unrealistic outcome. When simulating interest rates, the CIR model's predictions showed greater fluctuations for larger interest rates compared to rates near zero. In contrast, the empirical data displayed less volatility, a characteristic captured more accurately by the Vasicek model.

To evaluate the models, summary statistics together with the RMSE were employed, and

Vasicec model demonstrated a better fit in the dataset. However, the choice between the two models should be made with caution, taking into account the specific interest rate characteristics and context. If interest rates significantly deviate from zero, the Vasicek model may be preferred due to its tractability and the availability of a closed-form solution for derivatives. Nonetheless, when interests rates are close to zero, the Vasicek model's potential for negative rates can lead to illogical results. Moreover, both models struggle with complex term structures, as a smaller number of parameters may hinder accurate calibration to market data. In such cases, time-dependent models that can perfectly fit the current term structure might be more suitable. Ultimately, the choice of model should align with the practical needs and characteristics of the interest rates under consideration.

5.2 Recommendations

This study primarily centered on calibrating equilibrium single-factor models through Ordinary Least Squares (OLS) estimation method. To further enhance the understanding of interest rate modeling, it is advisable for future research to explore and compare alternative estimation methods. Additionally, future studies could delve into assessing the effectiveness of time-dependent models and multi-factor models in capturing the intricate dynamics for interest rate models in Kenyan context. Such investigations could provide valuable insights and contribute to a more comprehensive understanding of interest rate modeling and its applications

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