ENUMERATION OF PLANE AND d-ARY TREE-LIKE STRUCTURES

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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DEDICATION

To my family and friends …

ABSTRACT

Trees are connected graphs which do not have loops, multiple edges and cycles. A variety of trees such as binary trees, ordered trees, d-ary trees, Cayley trees and noncrossing trees have been studied at length. Tree-like structures such as cacti and Husimi graphs have the properties of trees where we consider blocks of the structures instead of vertices. Plane Husimi graphs, plane cacti and plane oriented cacti have been enumerated with regards to leaves, number of blocks and block types. However, there is no literature on the study of plane tree-like structures according to root degree and degree sequence. Moreover, d-ary tree-like structures have not been enumerated at all . In this work, we have enumerated plane Husimi graphs, plane cacti and plane oriented cacti according to the degree of the root and outdegree sequences. We have also enumerated bicoloured plane tree-like structures with regards to number of vertices, blocks and block types. Finally, we have introduced and enumerated d-ary Husimi graphs, cacti and oriented cacti with given indegree sequence, number of leaves, blocks and block types. To obtain our results we have used symbolic method to obtain generating functions for tree-like structures, used Lagrange Inversion formula and Lagrange Bürmann to extract the coefficients of the variables in the generating functions and in some instance, we constructed a bijection. The results of this study will add to literature in this area of study.

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CHAPTER 1

INTRODUCTION

Combinatorics is an area of mathematics which primarily deals with counting. The counting objects include sets, graphs, partitions among others. The branch of combinatorics that deals with obtaining exact formulas of these structures with a collection of parameters is called *enumerative combinatorics.* Our research is in enumerative combinatorics and our counting objects are tree-like structures which we define in the next section. In this work, we have obtained explicit formulas of these combinatorial objects and in one instance we have constructed a bijection.

1.1 Basic concepts

In this section, we illustrate basic mathematical concepts which we require to present our results.

1.1.1 Graph theoretic concepts

Concepts in this section can be found in the book by Diestel [5].

A *graph* G constitutes a pair $(V(G), E(G))$ with $V(G)$ a collection of vertices and $E(G)$ a collection of edges of G. A *loop* is an edge from a vertex into itself. *Multiple edges* are at least two edges that share initial vertex and terminal vertex. A *simple graph* lacks both loops and multiple edges. The number of vertices (resp. edges) of a graph is called the *order* (resp. *size*) of the graph. A *path* in a graph is a finite or infinite sequence of edges which connect a given number of vertices in the graph. A *cycle* is a path that starts at a given vertex and ends at the same vertex. Consider a graph in which there is a path between any pair of distinct vertices. Then such a graph is said to be *connected*. A *degree* of a vertex in a graph is the number of edges incident to the vertex. A *subgraph* H of a graph G is a graph with some or all vertices of G and edges are some or all edges of G. A *tree* is a connected graph without cycles, loops and multiple edges. Figure 1.1 is a depiction of a tree on 8 vertices.

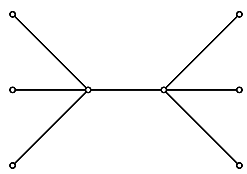

Figure 1.1: Tree on 8 vertices.

Given a tree T, a *subtree* of T is a subgraph of T which is also a tree. A *leaf* in a tree is a vertex of degree 1 whereas a non-leaf vertex is an *internal vertex*. A collection of trees is a *forest*. A *noncrossing tree* is a connected acyclic graph with edges that do not cross inside the circle and vertices on its boundary. See Figure 1.2 for an example of a noncrossing tree.

If we designate a vertex of a tree then we have a *rooted tree*. Given a vertex u in a plane tree, all the vertices at a lower level which are connected to u, are

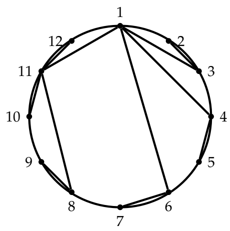

Figure 1.2: Noncrossing tree.

said to be *children* of u. A *plane tree* (or *ordered tree*) is a rooted tree drawn in the plane such that all children of internal vertices are ordered. In Figure 1.3, we have two different plane trees. (See [19] for more details.) In a plane tree, the

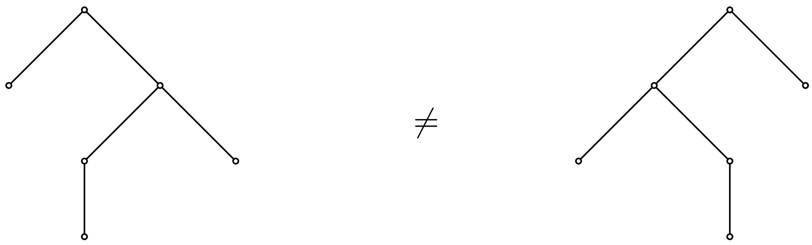

Figure 1.3: Plane trees on 6 vertices

number of children of the vertex is its *degree* and, *degree sequence* is the monotonic non-increasing sequence of the vertex degree of the tree. The degree sequence of the plane trees in Figure 1.3 is $2, 2, 1, 0, 0, 0$. A *d-ary tree* is a plane tree in which every internal vertex has no more than d children. 2-ary and 3-ary trees are called binary and ternary trees respectively. In Figure 1.4, we show a ternary tree on 10 vertices.

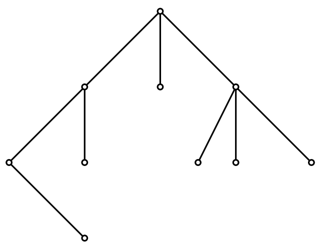

Figure 1.4: Ternary trees on 10 vertices.

The number of d-ary trees on $dn+1$ vertices is the generalised Catalan number,

$$\frac{1}{n+1}\binom{dn}{n-1}.$$

(See [19]). Some of the properties of trees are connectedness and cycle freeness. Structures which satisfy these properties are called *tree-like structures*. A *cutpoint* of a connected graph G is a vertex whose removal will disconnect G. A graph which has no cutpoint is said to be *2-connected*. A *block* in a simple graph is a maximal 2-connected subgraph. A *complete graph* is a simple graph in which every vertex is adjacent to all other vertices. In 1950, Kodi Husimi [11] introduced *Husimi graphs.* These are connected graphs whose blocks are complete graphs. If the blocks are cycles or polygons then we obtain *cacti* (singular *cactus*). Cacti were first studied by Harary and Unlenbeck [10]. The said authors called them 'Husimi trees'. *Oriented cacti* [18] are connected graphs where blocks are oriented cycles. So, Husimi graphs, cacti and oriented cacti are tree-like structures. If a tree-like structure is drawn in a plane such that blocks are ordered then we have a *plane tree-like structure*. Figure 1.5, shows a ternary cactus on 25 vertices. In a plane tree-like structure the number of blocks that are incident to a vertex is the *degree* of that vertex. A *leaf* is a non-root vertex which is incident to exactly one block. A *block child* is a block that is attached at a lower level of a particular vertex. The *outdegree* of vertex i is the number of block children of i. The *outdegree sequence* is an ordered sequence of the outdegrees of the vertices of the tree-like structure. If the outdegree of each vertex is at most d then we get a *d-ary tree-like structure*. A *bicoloured tree-like structure* is a tree-like structure in which the blocks are coloured using two colours such that blocks of the same colour are not incident to each other.

1.1.2 Generating functions and functional equations

Generating functions are vital in obtaining our results. Ordinary and exponential generating functions are used for counting unlabelled and labelled structures

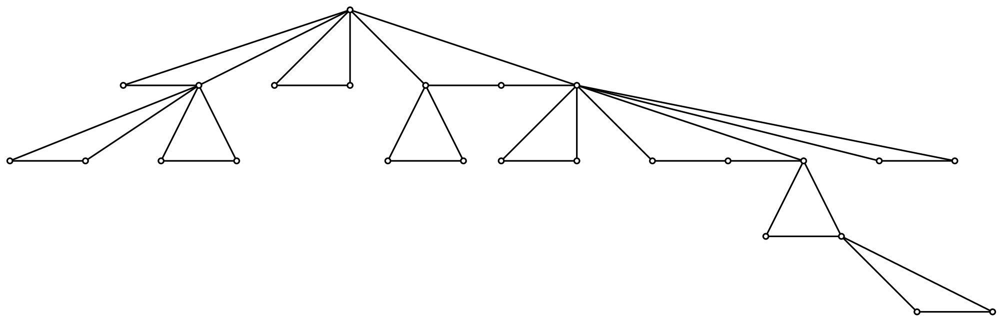

Figure 1.5: Ternary cactus on 25 vertices with 11 blocks.

respectively.

Definition 1.1.1. The *ordinary generating function* of the sequence $(s_0, s_1, \ldots)$, of integers, is $\sum_{i\geq 0} s_i x^i$ with its *exponential generating function* being $\sum_{i\geq 0} s_i \frac{x^i}{i!}$.

It is common practice to denote the coefficient of x^n in the generating function $g(x)$ by $[x^n]g(x)$. A *functional equation* is an equation expressing a function in terms of itself. We use the following theorems to extract the coefficient of x^n in a generating function $g(x)$ which takes the form $g(x) = x\psi(g(x))$.

Theorem 1.1.2 (Lagrange Inversion Formula, [19]). *Let $g(x)$ be a generating function that satisfies the functional equation $g(x) = x\psi(g(x))$, where $\psi(0) \neq 0$. Then, we have*

$$[x^n]g(x)^k = \frac{k}{n}[t^{n-k}]\psi(t)^n.$$

Theorem 1.1.3 (Lagrange-Bürmann Formula [8]). *Let $\psi(t)$ be a power series in t, not involving x. Then there is a unique power series $f = f(x)$ such that $f(x) = x\psi(f(x))$, and for any Laurent series $g(t)$, not involving x and for any integer $n \neq 0$ we have*

$$[x^n]g(f(x)) = \frac{1}{n}[t^{n-1}]\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right)\psi(t)^n.$$

Wilf's *generatingfunctionology* [21] is devoted to the study of generating functions and its applications.

1.1.3 Łukasiewicz words

Consider an alphabet $\mathcal{A}$ which consists of letters $l_0, l_1, \ldots$ and the empty word be denoted by 1. Let the weight of the letter l_i be $\varphi(l_i) = i - 1$. A word $w_1 w_2 \cdots w_m$ made of letters from $\mathcal{A}$ is a *Łukasiewicz word* if $\varphi(w_1) + \cdots + \varphi(w_j) \geq 0$ for $1 \leq j \leq m-1$ and $\varphi(w_1) + \cdots + \varphi(w_m) = -1$. Let $|\mathcal{A}| = n$ and let n_i be the multiplicity of letter l_i in a Łukasiewicz word. Then the number of *Łukasiewicz words* on $\mathcal{A}$ are

$$\binom{n}{n_0, n_1, \ldots} [19].$$

1.1.4 Basic identities

For integers $N, M, n, k \geq 0$, the following identities hold:[16]

(i)Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

(ii) Hockey-stick identity

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1},$$

(iii) Vandermonde identity

$$\sum_{k=0}^{n} \binom{N}{k} \binom{M}{n-k} = \binom{N+M}{n}, and$$

(iv) Multinomial theorem

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n = \sum_{n_1, n_2, \ldots, n_k \geq 0} \frac{n!}{n_1! n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where, $n_1 + n_2 + n_3 + \cdots + n_k = n$.

1.2 Statement of the problem

Various statistics concerning plane and d-ary trees have been investigated by both mathematicians and computer scientists. The parameters already studied are root degree, number of vertices, number of leaves, degree of a given vertex and degree of vertices at a given level. Plane tree-like structures have been enumerated by number of vertices, leaves, blocks and block types. However, there is no literature on the study of plane tree-like structures according to root degree and degree sequence. Moreover, d-ary tree-like structures have not been enumerated at all. In this work, we have obtained closed formulas for the number of plane Husimi graphs, cacti and oriented cacti according to root degree, total number of vertices of a given outdegree and outdegree/degree sequence. We have also enumerated bicoloured plane tree-like structures according to the number of vertices, blocks and block types. Finally, we have introduced and enumerated d-ary tree-like structures with regard to number of vertices, blocks, block types, outdegree sequence and number of leaves.

1.3 Objectives of the study

The aim of this work was to enumerate plane and d-ary tree-like structures according to the number of vertices, blocks, block types, root degree, outdegree sequence and number of leaves. Specifically, we have obtained:

i. Explicit formulas for the number of plane Husimi graphs, cacti and oriented cacti according to root degree and outdegree sequence.

ii. Explicit formulas for the number of bicoloured plane Husimi graphs, cacti and oriented cacti according to number of vertices, blocks and block types.

iii. Explicit formulas for the number of d-ary Husimi graphs, cacti and oriented cacti according to number of vertices, blocks, leaves, block types and outdegree sequence.

1.4 Methodology

To obtain our results, we have applied symbolic method to find generating functions for plane and d-ary tree-like Husimi graphs, cacti and oriented cacti. We then used Lagrange Inversion Formula and Lagrange-Bürmman Formula to extract the coefficients of the variables in the generating functions to obtain closed formulas. We have also constructed a bijection. Moreover, we have used direct proofs and built on the previous works of various authors.

1.5 Significance of the study

Plane and d-ary trees have been previously investigated by various mathematicians and computer scientists. However, no study has been done to enumerate plane tree-like structures using the parameters such as root degrees and outdegree sequences. Moreover, d-ary tree-like structures have not been enumerated. This study adds to the richly available literature. The results may be of importance to other areas of mathematics, physics and computer science.

CHAPTER 2

LITERATURE REVIEW

This chapter is dedicated to the review of literature on plane and d-ary trees as well as their tree-like counterparts. Trees and tree-like structures have been studied before [1, 2, 3, 4, 6, 7, 10, 14, 15, 17, 18]. Labelled trees on n vertices are counted by Cayley's formula, i.e n^{n-2} [1]. Dershowitz and Zaks [4] showed that the number of plane trees with n edges is given by,

$$C_n = \frac{1}{n}\binom{2n-2}{n-1},$$

which is the famous Catalan number. R. Stanley in [19] later showed that the number of d-ary trees with n internal vertices is given by

$$\frac{1}{n+1}\binom{dn}{n-1}.$$

In an attempt to generalize the concept of trees and solving counting problems arising from physics, Japanese physicist Kodi Husimi introduced Husimi graphs back in 1950 (See [11]). The author showed that if $(n_2, n_3, \ldots)$ is a sequence of non-negative integers satisfying the condition:

$$n = \sum_{i \geq 2}(i-1)n_i + 1, \tag{2.1}$$

then the number of Husimi graphs having n_i blocks of size i is

$$\frac{(n-1)!n^{k-1}}{\prod_{i\geq 2}((i-1)!)^{n_i} n_i!}, \tag{2.2}$$

where k is the total number of blocks. Husimi proved the formula by first obtaining a recurrence equation satisfied by the graphs and solving the recurrence relation. Mayer [13] provided a direct proof and Leroux [12] used generating functions to prove the formula. Summing over all n_i in (2.2), we find that there are

$$\left\{ {n-1 \atop k} \right\} n^{k-1}$$

Husimi graphs on $n \geq 1$ nodes with k blocks, where $\left\{ {m \atop r} \right\}$ is the Stirling's number of the second kind, which counts set partition.

In 1953, Harary and Unlenbeck in [10] introduced cacti and enumerated them according to number of blocks and sizes. Collin Springer in [18] then enumerated oriented cacti, again by the number of blocks and block sizes.

Okoth, in his PhD thesis [14], introduced coloured Husimi graphs, cacti and oriented cacti. These are structures whose blocks are assigned two different colours such that incident blocks are assigned different colours. He showed that these structures were in bijections with some families of set partitions introduced by Teufl and Wagner [20].

In the same thesis [14], he also introduced noncrossing tree-like structures and enumerated them according to number of blocks and block sizes. He also obtained counting formulas for these structures such that blocks are coloured using only two colours and incident blocks do not receive the same colour. He called them *bicoloured noncrossing Husimi graphs*. To obtain his results, the aforementioned author used generating function approach.

In 2021, Okoth in his paper [15] studied plane tree-like structures and found explicit formulas for plane Husimi graphs, cacti and oriented cacti with given number of blocks, block sizes and leaves. He mainly used symbolic methods to find generating functions for such structures and then used Lagrange Inversion Formula to extract the coefficients of the variables in the generating function.

Plane Husimi graphs had not been counted with root degree and outdegree sequence as the parameters of enumeration. Moreover, d-ary tree-like structures have not been enumerated by number of vertices, leaves, blocks, block types, root degree and outdegree sequence. In this work, we have built on the latest studies of Okoth [15] and obtained closed formulas for plane tree-like structures according to root degree and outdegree sequence. We have also enumerated d-ary tree-like structures according to number of blocks, block types, leaves, root degree and outdegree sequences.

CHAPTER 3

ENUMERATION OF PLANE TREE-LIKE STRUCTURES

We present our results on plane tree-like structures. We have obtained closed formulas for plane Husimi graphs, cacti and oriented cacti with a given root degree, total number of vertices of a given outdegree and outdegree sequence. We have also obtained closed formulas for the number of bicoloured plane tree-like structures according to number of vertices, blocks and block types. We begin by enumerating plane tree-like structures by outdegree sequence.

3.1 Counting plane tree-like structures by outdegree sequences

In this section, plane Husimi graphs, cacti and oriented cacti are enumerated by number of vertices of a given degree, total number of vertices of a given outdegree and outdegree sequences. We also obtain a formula counting these graphs with a given root degree. We start by proving the following important lemma.

Lemma 3.1.1. *Let $n, k \geq 1$ and let $n_1, n_2, \ldots$ be non-negative integers such that $n_1 + n_2 + \cdots = k$ and $n_1 + 2n_2 + \cdots = n - 1$. Then,*

$$\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{k!}{n_1!n_2!\cdots} = \binom{n-2}{k-1}.$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{1}{n_1!n_2!\cdots} &= [x^{n-1}y^k]\prod_{i\geq 1}\left(\sum_{j\geq 0}\frac{x^{ij}y^j}{j!}\right) \\
&= [x^{n-1}y^k]\prod_{i\geq 1}\exp(x^i y) \\
&= [x^{n-1}y^k]\exp(y(x+x^2+\cdots)) \\
&= [x^{n-1}y^k]\exp\left(\frac{xy}{1-x}\right) \\
&= [x^{n-1}y^k]\sum_{i\geq 0}\frac{x^i y^i (1-x)^{-i}}{i!} \\
&= [x^{n-1}]\frac{x^k(1-x)^{-k}}{k!}.
\end{aligned}$$

By binomial theorem, we have

$$\begin{aligned}
\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{1}{n_1!n_2!\cdots} &= [x^{n-k-1}]\frac{1}{k!}\sum_{i\geq 0}\binom{-k}{i}(-x)^i \\
&= [x^{n-k-1}]\frac{1}{k!}\sum_{i\geq 0}\binom{k+i-1}{i}x^i \\
&= \frac{1}{k!}\binom{n-2}{n-k-1}.
\end{aligned}$$

□

Theorem 3.1.2. *There are*

$$\frac{1}{n}\binom{n}{d_0, d_1, \ldots, d_k}\binom{n-2}{k-1} \tag{3.1}$$

plane Husimi graphs on n vertices with k blocks and exactly d_i vertices of outdegree i.

Proof. Consider any plane Husimi graph H of order n with k blocks such that there are n_i blocks of size $i \geq 2$. Label the vertices of the graph with integers $1, 2, \ldots n$ such that node i is the i^{th} node visited when H is traversed in preorder (i.e visit the root, left most subtree, second most subtree, etc). Let d_i be the out-degree of vertex i.

Let A be the set of words of length n. Also, let B the set of Łukasiewicz words of length n. In the book [19], R. Stanley constructed a bijection $\phi : A \times [n] \longrightarrow B \times [r]$ by means of plane forests with r components where $[n] := \{1, 2, \ldots, n\}$. Setting $r = 1$, we obtain the necessary result since we are dealing with tree-like structures and not forest of tree-like structures. Here, the set A is the set of words $x_{d_1}x_{d_2}\cdots x_{d_n}$ where d_i is the degree of vertex i.

From the bijection, we have

$$n|A| = \binom{n}{d_0, d_1, \ldots, d_k}$$

since $d_j = 0$ for all $j > k$. Making use of Lemma 3.1.1 to sum over all block types, the number of plane Husimi graphs on n vertices with k blocks is thus

$$|A|\binom{n-2}{k-1} = \frac{1}{n}\binom{n}{d_0, d_1, \ldots, d_k}\binom{n-2}{k-1}.$$

□

Corollary 3.1.3. *The total number of vertices of outdegree $i \geq 0$ over all plane Husimi graphs on $n \geq 1$ vertices with k blocks is*

$$\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}. \tag{3.2}$$

Proof. We first obtain the sum,

$$\begin{aligned}
\sum_{\substack{d_1+d_2+\cdots=n \\ d_1+2d_2+\cdots=k \\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{d_1!d_2!\cdots} &= [x^k y^n]\left(\sum_{j\geq 1}\frac{jx^{ij}y^j}{j!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right) \\
&= [x^k y^n]x^i y\left(\sum_{j\geq 1}\frac{x^{i(j-1)}y^{(j-1)}}{(j-1)!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right) \\
&= [x^k y^n]x^i y\left(\sum_{j\geq 0}\frac{x^{ij}y^j}{j!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right) \\
&= [x^k y^n]x^i y\exp(x^i y)\prod_{\substack{m\geq 1\\ m\neq i}}\exp(x^m y) \\
&= [x^k y^n]x^i y\prod_{m\geq 1}\exp(x^m y) \\
&= [x^k y^n]x^i y\exp(y(x+x^2+\cdots)) \\
&= [x^{k-i}y^{n-1}]\exp\left(\frac{xy}{1-x}\right) \\
&= [x^{k-i}y^{n-1}]\sum_{i\geq 0}\frac{x^i y^i(1-x)^{-i}}{i!} \\
&= [x^{k-i}]\frac{x^{n-1}(1-x)^{-(n-1)}}{(n-1)!}.
\end{aligned}$$

By binomial theorem, we have

$$\sum_{\substack{d_1+d_2+\cdots=n\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{d_1!d_2!\cdots} = [x^{k-i-n+1}]\frac{1}{(n-1)!}\sum_{j\geq 0}\binom{-(n-1)}{j}(-x)^j$$

$$= [x^{k-i-n+1}]\frac{1}{(n-1)!}\sum_{j\geq 0}\binom{n+j-2}{j}x^j$$

$$= \frac{1}{(n-1)!}\binom{k-i-1}{k-i-n+1}.$$

Let d_i be the number of vertices of outdegree i. Now, we have the total number of vertices of outdegree $i > 0$ in plane Husimi graphs on n vertices with k blocks as:

$$\sum_{d_0\geq 0}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{n}\binom{n}{d_0,d_1,\ldots,d_k}\binom{n-2}{k-1}$$

$$= \sum_{d_0\geq 0}\frac{1}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}}\frac{d_i}{d_1!d_2!\cdots d_k!}$$

$$= \sum_{d_0\geq 0}\frac{1}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0-1)!}\binom{k-i-1}{n-d_0-2}$$

$$= \sum_{d_0\geq 0}\binom{n-1}{d_0}\binom{k-i-1}{n-d_0-2}\binom{n-2}{k-1}$$

$$= \binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.$$

The last equality follows by Vandermonde identity. Next, we obtain the number of vertices of outdegree 0, i.e., we get the sum

$$\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_0}{n}\binom{n}{d_0,d_1,\ldots,d_k}\binom{n-2}{k-1}$$

$$\begin{aligned}
&= \frac{d_0}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{1}{d_1!d_2!\cdots d_k!}\\
&= \binom{n-1}{d_0-1}(n-d_0)!\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0)!}\binom{k-1}{n-d_0-1}\\
&= \binom{n-1}{d_0-1}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}.
\end{aligned}$$

The formula follows by summing over all d_0 making use of Vandermonde identity. $\square$

Corollary 3.1.4 was also proved by Okoth [15] using generating functions.

Corollary 3.1.4. *There are*

$$\frac{1}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1} \tag{3.3}$$

plane Husimi graphs on n vertices with k blocks and d_0 leaves.

Proof. We sum over all d_i for $i=1,2,\ldots$ in Equation (3.1):

$$\sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0,d_2\geq 0,\ldots,d_k\geq 0}} \frac{1}{n}\binom{n}{d_0,d_1,\ldots,d_k}\binom{n-2}{k-1}$$
$$= \frac{1}{n}\binom{n}{d_0} \sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0,d_2\geq 0,\ldots,d_k\geq 0}} \binom{n-d_0}{d_1,d_2,\ldots,d_k}\binom{n-2}{k-1}$$
$$= \frac{(n-d_0)!}{n}\binom{n}{d_0}\binom{n-2}{k-1} \sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0,d_2\geq 0,\ldots,d_k\geq 0}} \frac{1}{d_1!d_2!\cdots d_k!}.$$

From the proof of Lemma 3.1.1, we have the sum as:

$$\frac{(n-d_0)!}{n}\binom{n}{d_0}\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0)!}\binom{k-1}{n-d_0-1}$$
$$= \frac{1}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}.$$

This completes the proof. □

The total number of leaves in plane Husimi graphs on n vertices with k blocks is thus

$$\sum_{d_0=1}^{n-k}\frac{d_0}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}$$
$$= \sum_{d_0=1}^{n-k}\binom{n-1}{d_0-1}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}$$
$$= \binom{n+k-2}{n-2}\binom{n-2}{k-1}.$$

Lemma 3.1.5. *The number of plane Husimi graphs on n vertices with k blocks such that the root has degree r is given by*

$$\frac{r}{k}\binom{n+k-r-2}{k-r}\binom{n-2}{k-1}. \tag{3.4}$$

Proof. Let T be a plane Husimi graph on n vertices such that the root has degree r. Using Depth First Search (DFS), we label the vertices of the graph with integers

$1, 2, \ldots, n$ such that the root is labelled 1. Let the degree of vertex i be r_i. Then $r + r_2 + r_3 + \cdots + r_n = k$. The number of nonnegative integer solutions of the equation $r_2 + r_3 + \cdots + r_n = k - r$ is $\binom{n+k-r-2}{k-r}$. Moreover, there are exactly r permutations of the total k cyclic permutations of $r_2, r_3, \ldots, r_n$ which are valid degree sequences.

Since there are a total of $\binom{n-2}{k-1}$ choices for block types (see Lemma 3.1.1) if there are n vertices in the plane tree-like structure with k blocks then the result follows.

□

We provide alternative proof for Lemma 3.1.5 based on generating functions and making use of Lagrange-Bürmann Formula proved in [8].

Alternative proof of Lemma 3.1.5. Let $P(x)$ be the generating function for plane Husimi graphs, where x marks a vertex. Let y_i mark blocks of size i. Then

$$P(x) = x + x \sum_{i \geq 1} y_{i+1} P^i + x \left(\sum_{i \geq 1} y_{i+1} P^i \right)^2 + \cdots = \frac{x}{1 - \sum_{i \geq 1} y_{i+1} P^i}.$$

Thus the generating function for plane Husimi graphs with root degree r is $x \left(\sum_{i \geq 1} y_{i+1} P^i \right)^r$. We remain to extract the coefficient of x^n in $x \left(\sum_{i \geq 1} y_{i+1} P^i \right)^r$, By Lagrange-Bürmann formula (Theorem 1.1.3), we have

$$\begin{aligned}
[x^n]x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r &= [x^{n-1}](\sum_{i\geq 1} y_{i+1}P^i)^r \\
&= \frac{1}{n-1}[t^{n-2}]r\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right) \\
&\qquad \left(\frac{x}{1-\sum_{i\geq 1} y_{i+1}t^i}\right)^{n-1} \\
&= \frac{r}{n-1}[t^{n-2}]\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right)\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell} \\
&\qquad \left(\sum_{i\geq 1} y_{i+1}t^i\right)^{\ell} \\
&= \frac{r}{n-1}[t^{n-2}]\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right) \\
&= \frac{r}{n-1}[t^{n-2}]\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r+\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell} \\
&= \frac{r}{n-1}\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}(n-1)[t^{n-1}]\frac{1}{r+\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell} \\
&= \sum_{\ell\geq 0}\frac{r}{r+\ell}\binom{n+\ell-2}{\ell}[t^{n-1}]\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell}.
\end{aligned}$$

We thus have,

$$[x^n]x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r = \sum_{\ell\geq 0}\frac{r}{r+\ell}\binom{n+\ell-2}{\ell}\sum_{\substack{n_2+n_3+\cdots=r+\ell \\ n_2+2n_3+\cdots=n-1}}\frac{(r+\ell)!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\ldots}.$$

So, the number of plane Husimi graphs on n vertices, k blocks and root degree r such that there are n_i blocks of size i is given by

$$r\binom{n+k-r-2}{k-r}\frac{(k-1)!}{n_2!n_3!\ldots}$$

where $n_2+n_3+\cdots=k$. By the proof of Lemma 3.1.1, we have that the number of plane Husimi graphs on n vertices, k blocks and root degree r is

$$
\begin{aligned}
r(k-1)!\binom{n+k-r-2}{k-r} \sum_{n_2+n_3+\cdots=k} \frac{1}{n_2!n_3!\dots}
&= r(k-1)!\binom{n+k-r-2}{k-r} \cdot \frac{1}{k!}\binom{n-2}{n-k-1} \\
&= \frac{r}{k}\binom{n+k-r-2}{k-r}\binom{n-2}{k-1}.
\end{aligned}
$$

□

Corollary 3.1.6. *There are a total of*

$$\frac{1}{n}\binom{n+k-1}{k}\binom{n-2}{k-1}$$

plane Husimi graphs on n vertices with k blocks.

Proof. We sum over all r in Equation (3.4):

$$
\begin{aligned}
\sum_{r=1}^{k} \frac{r}{k}\binom{n+k-r-2}{k-r}\binom{n-2}{k-1} &= \sum_{r=1}^{k} \frac{r}{k}\binom{n+k-r-2}{n-2}\binom{n-2}{k-1} \\
&= \sum_{i=n-2}^{n+k-3} \frac{n+k-i-2}{k}\binom{i}{n-2}\binom{n-2}{k-1} \\
&= \sum_{i=n-2}^{n+k-3} \frac{n+k-1}{k}\binom{i}{n-2}\binom{n-2}{k-1} - \sum_{i=n-2}^{n+k-3} \frac{(i+1)(n-1)}{k(n-1)}\binom{i}{n-2}\binom{n-2}{k-1} \\
&= \frac{n+k-1}{k}\sum_{i=n-2}^{n+k-3}\binom{i}{n-2}\binom{n-2}{k-1} - \sum_{i=n-2}^{n+k-3} \frac{n-1}{k}\binom{i+1}{n-1}\binom{n-2}{k-1} \\
&= \frac{n+k-1}{k}\sum_{i=n-2}^{n+k-3}\binom{i}{n-2}\binom{n-2}{k-1} - \frac{n-1}{k}\sum_{j=n-1}^{n+k-2}\binom{j}{n-1}\binom{n-2}{k-1} \\
&= \frac{n+k-1}{k}\binom{n+k-2}{n-1}\binom{n-2}{k-1} - \frac{n-1}{k}\binom{n+k-1}{n}\binom{n-2}{k-1} \\
&= \frac{1}{k}\binom{n+k-1}{n}\binom{n-2}{k-1} \\
&= \frac{1}{n}\binom{n+k-1}{k}\binom{n-2}{k-1}.
\end{aligned}
$$

□

We now obtain the following result:

Theorem 3.1.7. *Let $n, r \geq 1$ and $i \geq 0$. Then the total number of vertices with outdegree i among all plane Husimi graphs on n vertices with root degree r and k blocks is given by*

$$\begin{cases} r\binom{n+k-r-i-3}{n-3}\binom{n-2}{k-1}, & \textit{when} \quad i \neq r, \\ \\ r\binom{n+k-2r-3}{n-3}\binom{n-2}{k-1} + \frac{r}{k}\binom{n+k-r-2}{n-2}\binom{n-2}{k-1}, & \textit{when} \quad i = r. \end{cases}$$

Proof. We consider the two cases:

Case 1: Let $i \neq r$. Let T be a plane Husimi graph on n vertices with k blocks such that the root has degree r and a given vertex u has outdegree i. Let the outdegree of the remaining vertices be $d_1, d_2, \ldots, d_{n-2}$. Again these outdegrees of T are arranged as one traverses the Husimi graph by DFS. The total number of nonnegative integer solutions of the equation $d_1 + d_2 + \cdots + d_{n-2} = k - r - i$ is $\binom{n+k-r-i-3}{n-3}$ [19]. This proves the result.

Case 2: For $i = r$, we need to note that the roots are also counted. Thus the result follows by adding the result of Case 1 and Equation (3.4). $\square$

Corollary 3.1.8. *The total number of vertices of degree $i \geq 1$ over all plane Husimi graphs on $n \geq 1$ vertices with k blocks is*

$$\frac{n+k-1}{k}\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.$$

Proof. The desired formula is the sum of the total number of non-root vertices of outdegree $i-1$ and the number of roots of degree i in plane Husimi graphs on n vertices with k blocks. By Equations (3.2) and (3.4), we have the required formula as:

$$
\begin{aligned}
&\left[\binom{n+k-i-1}{n-2} - \frac{i-1}{k}\binom{n+k-i-1}{n-2} + \frac{i}{k}\binom{n+k-i-2}{n-2}\right]\binom{n-2}{k-1} \\
&= \left[\frac{k-i+1}{k}\binom{n+k-i-1}{n-2} + \frac{i}{k}\binom{n+k-i-2}{n-2}\right]\binom{n-2}{k-1} \\
&= \frac{n+k-1}{k}\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.
\end{aligned}
$$

□

3.2 Bicoloured plane tree-like structures

We recall from Subsection 1.1.1 that a *bicoloured plane tree-like structure* is a tree-like structure whose blocks are coloured using two colours such that no blocks of the same colour are incident to one another. A *2-colourable plane tree-like structure* is a structure with root degree at most 2 and the rest of the vertices with at most degree 1. This makes it possible to colour the blocks using two colours. We start by enumerating 2-colourable plane tree-like structures with roots of degree 1.

Proposition 3.2.1. *The number of 2-colourable plane Husimi graphs on n vertices and k blocks such that the root of graph has degree* 1 *and there are n_j blocks of size j is given by*

$$\frac{1}{n}\binom{n}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!}. \tag{3.5}$$

Proof. Let $B(x)$ be the generating function for 2-colourable plane Husimi graphs with root degree 1 (or 0). Since each vertex is to have degree less than or equal to 2, the generating function satisfies $B(x) = x(1+\sum_{i\geq 1} y_{i+1}B^i)$. By the Lagrange

Inversion Formula [19], we have

$$
\begin{aligned}
[x^n]B(x) &= \frac{1}{n}[t^{n-1}](1+\sum_{i\geq 1} y_{i+1}t^i)^n \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{n}{k}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^k \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{n}{k}(y_2t+y_3t^2+\cdots)^k \\
&= \frac{1}{n}\sum_{k\geq 0}\binom{n}{k}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{k!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\cdots}
\end{aligned}
$$

Therefore, there are

$$\frac{1}{n}\binom{n}{k}\frac{k!}{\prod_{j\geq 2} n_j!}$$

such 2-colourable plane Husimi graphs. □

Corollary 3.2.2. *The number of 2-colourable plane Husimi graphs with root of degree* 1 *is given by the Narayana number,*

$$\frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}. \tag{3.6}$$

Proof. By the proof of Lemma 3.1.1, we have

$$\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{k!}{n_2!n_3!\cdots}=\binom{n-2}{k-1}.$$

Formula (3.6) thus follows immediately from Equation (3.5) by summing over all n_j. □

Summing over all k in Equation (3.6), we find another combinatorial structure counted by the Catalan number.

Corollary 3.2.3. *The number of 2-colourable plane cacti on n vertices and k cycles such that the root degree is* 1*, and n_j cycles of size j is given by*

$$\frac{1}{n}\binom{n}{k}\frac{k!}{\prod_{j\geq 3} n_j!}.$$

Proof. The result follows from Equation (3.5) by noting that we can convert a complete graph to a cycle by deleting all edges except the boundary ones. □

Corollary 3.2.4. *The number of 2-colourable plane oriented cacti with root of degree* 1*, having n vertices and k blocks, n_j of which are of size j is given by*

$$\frac{2^{k-n_2}(n-1)!}{(n-k)!\prod_{j\geq 3} n_j!}. \tag{3.7}$$

Proof. The proof follows since every edge in a cycle of size at least three has two orientations. □

Corollary 3.2.5. *The number of 2-colourable plane oriented cacti on n vertices such that root is of degree* 1 *is given by*

$$\sum_{k\geq 1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{2^{k-n_2}(n-1)!}{(n-k)!\prod_{j\geq 3} n_j!}.$$

Proof. The result follows by summing over all n_j and over all k in Equation (3.7). □

Proposition 3.2.6. *The number of 2-colourable plane Husimi graphs on n vertices and k blocks such that the root of the graph has degree* 2 *and there are n_j blocks of size j is given by*

$$\frac{2}{n+1}\binom{n+1}{k}\frac{k!}{\prod_{j\geq 2} n_j!}. \tag{3.8}$$

Proof. Let $C(x)$ be the generating function for 2-colourable plane Husimi graphs with root degree 2. The generating function is obtained by merging two such graphs with root degree 1, which satisfies $C(x) = \dfrac{B(x)^2}{x}$, where $B(x)$ is the generating function of a 2-colourable plane Husimi graphs with root of degree 1 By the Lagrange inversion formula, we obtain

$$\begin{aligned}
[x^n]C(x) &= [x^n]\frac{B^2(x)}{x} \\
&= [x^{n+1}]B^2(x) \\
&= \frac{2}{n+1}[t^{n-1}]\left(1+\sum_{i\geq 1} y_{i+1}t^i\right)^{n+1} \\
&= \frac{2}{n+1}[t^{n-1}]\sum_{k\geq 0}\binom{n+1}{k}(y_2 t + y_3 t^2 + \cdots)^k \\
&= \frac{2}{n+1}\sum_{k\geq 0}\binom{n+1}{k}\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}}\frac{k! y_2^{n_2} y_3^{n_3} \ldots}{n_2! n_3! \cdots}.
\end{aligned}$$

Therefore, the number of the required plane Husimi graphs is

$$\frac{2}{n+1}\binom{n+1}{k}\frac{k!}{\prod_{j\geq 2} n_j!}.$$

□

Corollary 3.2.7. *The number of 2-colourable plane Husimi graphs on n vertices with k blocks, n_j of which have size j is given by*

$$\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\frac{k!}{\prod_{j\geq 2} n_j!}. \tag{3.9}$$

Proof. We get the required formula by adding Equations (3.5) and (3.8), i.e.

$$\begin{aligned}
&\left[\frac{1}{n}\binom{n}{k}+\frac{2}{n+1}\binom{n+1}{k}\right]\frac{k!}{\prod\limits_{j\geq 2} n_j!}\\
&=\left[\frac{1}{n}\binom{n}{k}+\frac{2}{n-k+1}\binom{n}{k}\right]\frac{k!}{\prod\limits_{j\geq 2} n_j!}\\
&=\frac{n-k+1+2n}{n(n-k+1)}\binom{n}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!}\\
&=\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!}.
\end{aligned}$$

□

Summing over all n_j in Equation (3.9), we find that there are

$$\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}$$

2-colourable plane Husimi graphs on n vertices with k blocks and thus the total number of 2-colourable plane Husimi graphs on n vertices is

$$\sum_{k\geq 1}\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}.$$

Using a similar argument, there are

$$\sum_{k\geq 1}\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}$$

and

$$\sum_{k\geq 1}\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}2^{k-n_2}$$

2-colourable cacti and 2-colourable oriented cacti respectively on n vertices.

Corollary 3.2.8. *The number of bicoloured plane Husimi graph on n vertices with k blocks such that n_j blocks are of size j is given by*

$$\frac{3n-k+1}{n-k+1}\cdot\frac{2}{n}\binom{n}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!}. \tag{3.10}$$

Proof. There are two ways of colouring a block and one way for colouring the remaining blocks. Thus, the required formula is twice Equation (3.9). □

Corollary 3.2.9. *The number of bicoloured plane Husimi graphs on n vertices with k blocks is given by*

$$\frac{3n-k+1}{n-k+1}\cdot\frac{2}{n}\binom{n}{k}\binom{n-2}{k-1}.$$

Proof. We sum over all n_j in Equation (3.10). □

Also, there are

$$\sum_{k\geq 1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{3n-k+1}{n-k+1}\cdot\frac{2}{n}\binom{n}{k}\frac{k!}{(n-k)!\prod_{j\geq 3}n_j!}$$

and

$$\sum_{k\geq 1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{3n-k+1}{n-k+1}\cdot\frac{1}{n}\binom{n}{k}\frac{2^{k-n_2+1}\cdot k!}{(n-k)!\prod_{j\geq 3}n_j!}$$

bicoloured cacti and bicoloured oriented cacti respectively on n vertices.

CHAPTER 4

ENUMERATION OF d-ARY TREE-LIKE STRUCTURES

In this chapter, we present our results on d-ary tree-like structures. We obtain closed formulas for the number of d-ary Husimi graphs, d-ary cacti and d-ary oriented cacti according to number of vertices, blocks, block types, outdegree sequence and number of leaves.

4.1 Enumeration by blocks and block types

We begin by getting the number of d-ary Husimi graphs on a given number of vertices.

Theorem 4.1.1. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number $dHG_n(n_2, n_3, \ldots)$ of d-ary Husimi* *graphs on n vertices having n_j blocks of size j is*

$$dHG_n(n_2, n_3, \dots) = \frac{1}{n}\binom{dn}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!} \tag{4.1}$$

where k is the total number of blocks.

Proof. Let x mark a vertex. Let $D(x)$ be the generating function for d-ary Husimi graphs. Let y_i denote the number of vertices in every block. The generating function $D(x)$ satisfies the functional equation $D(x) = x(1 + \sum_{i\geq 1} y_{i+1} D^i)^d$. By the Lagrange inversion formula (Theorem 1.1.2), we obtain

$$\begin{aligned}
[x^n]D(x) &= \frac{1}{n}[t^{n-1}]\left(1 + \sum_{i\geq 2} y_{i+1}t^i\right)^{dn} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{dn}{k}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^k \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{dn}{k}(y_2t + y_3t^2 + \cdots)^k \\
&= \frac{1}{n}\sum_{k\geq 0}\binom{dn}{k}\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}}\frac{k! y_2^{n_2} y_3^{n_3}\cdots}{n_2! n_3! \cdots}.
\end{aligned} \tag{4.2}$$

Thus, the required formula is

$$\frac{1}{n}\binom{dn}{k}\frac{k!}{\prod\limits_{j\geq 2} n_j!}.$$

□

Example 4.1.2. Consider a binary Husimi graph on 5 vertices with 2 blocks of type $(0, 2, 0, \dots)$ satisfying the coherence conditions. There are nine such graphs as given in Figure 4.1.

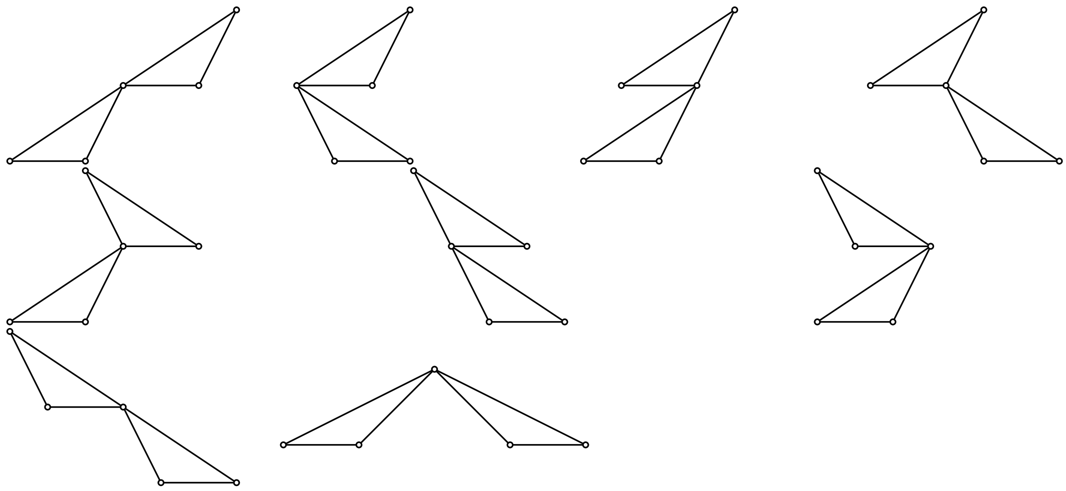

Figure 4.1: Binary Husimi graphs on 5 vertices with 2 blocks of type $(0, 2, 0, \ldots)$

Corollary 4.1.3. *The number of d-ary Husimi graphs on n vertices having k blocks is given by*

$$\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}. \tag{4.3}$$

Proof. From Equation (4.1), the required formula is given by

$$\frac{1}{n}\binom{dn}{k} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!}{n_2!n_3!\ldots}. \tag{4.4}$$

By Lemma 3.1.1, we have

$$\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!}{n_2!n_3!\ldots} = \binom{n-2}{k-1}. \tag{4.5}$$

Substituting Equation (4.5) in Equation (4.4), we find that the number of d-ary Husimi graphs on n vertices with k blocks is $\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}$. This completes the proof. $\square$

Summing over all k in Equation (4.3), we find that there are a total of

$$\frac{1}{n}\binom{(d+1)n-2}{n-1} \tag{4.6}$$

d-ary Husimi graphs on n vertices. Setting $d = 1$, we find that there are

$$\frac{1}{n}\binom{2n-2}{n-1}$$

unary Husimi graphs in which every vertex has outdegree 1 or 0. This is another manifestation of Catalan numbers. This formula also counts the number of plane trees on n vertices.

Lemma 4.1.4. *There is a bijection between the set of unary Husimi graphs on n vertices and plane trees on n vertices.*

Proof. To convert a unary Husimi graph on n vertices to a plane tree on n vertices we remove the edges between any two adjacent vertices on the same level and create edges between each internal vertex and its children. Conversely, by creating edges between vertices on the same level, we obtain a unary Husimi graph. See Figure 4.2 for the bijection. □

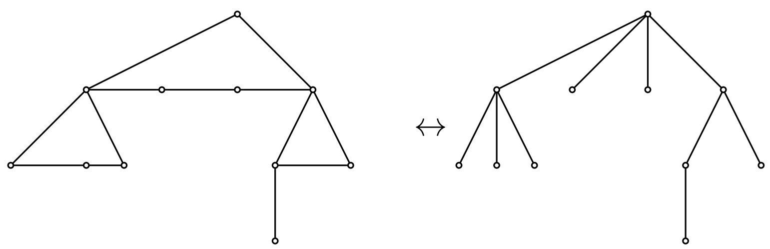

Figure 4.2: Bijection between unary Husimi graph and plane tree

Corollary 4.1.5. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number $dC_n(n_2, n_3, \ldots)$ of d-ary cacti on n vertices having n_j blocks of size j is*

$$dC_n(n_2, n_3, \ldots) = \frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 3} n_j!} \tag{4.7}$$

where k is the total number of blocks.

Proof. We can convert a complete graph to a cycle by deleting all the edges except the boundary ones. So the required equation follows from Equation (4.1) i.e, $dC_n(n_3, n_4, \ldots)$=$dHG_n(n_3, n_4, \ldots)$. $\square$

Summing over all n_j and k in Equation (4.7), we find the total number of d-ary cacti on n vertices as

$$\sum_{k\geq 1} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 3} n_j!}.$$

Since there are exactly two orientations for each block in a cactus, then there are

$$\sum_{k\geq 1} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{1}{n}\binom{dn}{k}\frac{2^{k-n_2}\cdot k!}{\prod_{j\geq 3} n_j!}$$

d-ary oriented cacti ($dOC_n(n_3, n_4, \ldots)$) on n vertices.

4.2 Enumeration by number of leaves

Theorem 4.2.1. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number of d-ary Husimi graphs on n vertices with ℓ leaves and having n_j blocks of size j is given by*

$$\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\binom{dm}{k}\frac{k!}{\prod_{j\geq 2} n_j!}, \tag{4.8}$$

where k is the number of blocks.

Proof. Let x (resp. u) mark a vertex (resp. leaf) in a d-ary Husimi graph. Also let y_i mark blocks of size i. Then the bivariate generating function for the number of d-ary Husimi graphs with given number of vertices and leaves is given by

$D(x,u) = x(u + (1 + \sum_{i\geq 2} y_{i+1} D^i)^d - 1)$. We obtain the coefficients of x^n and u^ℓ in the generating function. By Lagrange inversion formula [19], we have

$$
\begin{aligned}
[x^n u^\ell] D(x,u) &= \frac{1}{n}[u^\ell t^{n-1}] \left(u + \left(1 + \sum_{i\geq 1} y_{i+1} t^i\right)^d - 1 \right)^n \\
&= \frac{1}{n}[u^\ell t^{n-1}] \sum_{m=0}^{n} \binom{n}{m} u^m \left(\left(1 + \sum_{i\geq 1} y_{i+1} t^i\right)^d - 1 \right)^{n-m} \\
&= \frac{1}{n}\binom{n}{\ell} [t^{n-1}] \left(\left(1 + \sum_{i\geq 1} y_{i+1} t^i\right)^d - 1 \right)^{n-\ell} \\
&= \frac{1}{n}\binom{n}{\ell} [t^{n-1}] \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \left(1 + \sum_{i\geq 1} y_{i+1} t^i\right)^{dm} (-1)^{n-\ell-m} \\
&= \frac{1}{n}\binom{n}{\ell} [t^{n-1}] \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} (-1)^{n-\ell-m} \sum_{k=0}^{dm} \binom{dm}{k} \left(\sum_{i\geq 1} y_{i+1} t^i\right)^k \\
&= \frac{1}{n}\binom{n}{\ell} \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} (-1)^{n-\ell-m} \sum_{k=0}^{dm} \binom{dm}{k} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k! y_2^{n_2} y_3^{n_3} \cdots}{n_2! n_3! \cdots}.
\end{aligned}
\tag{4.9}
$$

Thus the required formula is

$$
\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\binom{dm}{k}\frac{k!}{\prod_{j\geq 2} n_j!},
$$

which completes the proof $\square$

By summing over all n_j in Equation (4.8), we obtain the following corollary:

Corollary 4.2.2. *There are*

$$
\frac{1}{n}\binom{n}{\ell}\binom{n-2}{k-1}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\binom{dm}{k}(-1)^{n-\ell-m}
$$

d-ary Husimi graphs on n vertices with k blocks and having ℓ leaves.

Corollary 4.2.3. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number of d-ary cacti with ℓ leaves and having n_j blocks of size j is given by*

$$\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}\binom{dm}{k}(-1)^{n-\ell-m}\frac{k!}{\prod_{j\geq 3} n_j!}$$

where k is the total number of blocks.

Proof. The result follows by noting that there is exactly one way of converting a d-ary Husimi graph to a d-ary cactus. □

Corollary 4.2.4. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number of d-ary oriented cacti with ℓ leaves and having n_j blocks of size j is given by*

$$\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}\binom{dm}{k}(-1)^{n-\ell-m}\frac{2^{k-n_2}.k!}{\prod_{j\geq 2} n_j!}$$

where k is the total number of blocks.

Proof. The result follows by noting that there are exactly two orientations for each block in a cactus. □

4.3 Enumeration by outdegree sequence

We prove the following theorem:

Theorem 4.3.1. *The number of d-ary Husimi graphs on n vertices with k blocks such that there are r_i vertices of outdegree i is*

$$\frac{1}{n}\binom{n}{r_0, r_1, \ldots, r_d}\binom{n-2}{k-1}\binom{d}{0}^{r_0}\binom{d}{1}^{r_1}\binom{d}{2}^{r_2}\cdots\binom{d}{d}^{r_d} \tag{4.10}$$

if $\sum_{i=0}^{d} ir_i = k$.

Proof. We consider a plane Husimi graph on n vertices with k blocks such that the maximum outdegree is d and that there are r_i vertices of outdegree i. The number of such graphs is given in Theorem 3.1.2. We can convert this graph to the required d-ary Husimi graph by selecting the positions of the block children for each vertex. If a vertex has j block children then there are $\binom{d}{j}$ positions for the block children in the d-ary Husimi graph. The result thus follows. □

By summing over all r_i satisfying the coherence conditions $r_0 + r_1 + \cdots + r_d = n$ and $r_1 + 2r_2 + \cdots = k$ we obtain the following corollary:

Corollary 4.3.2. *There are*

$$\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}$$

d-ary Husimi graphs on n vertices and k blocks.

Corollary 4.3.3. *The number of binary Husimi graphs on n vertices such that there are r_0 vertices with no block child, r_1 vertices with 1 block child and r_2 children with 2 block children is given by*

$$\frac{2^{n+r_1-2}}{n}\binom{n}{r_0, r_1, r_2}.$$

Proof. The result follows from Equation (4.10) by setting $d = 2$ and summing over all k, i.e., the desired result is

$$\sum_{k=1}^{n-1} \frac{1}{n} \binom{n}{r_0, r_1, r_2} \binom{n-2}{k-1} \binom{2}{0}^{r_0} \binom{2}{1}^{r_1} \binom{2}{2}^{r_2} \tag{4.11}$$

which we now simplify.

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{n} \binom{n}{r_0, r_1, r_2} \binom{n-2}{k-1} \binom{2}{1}^{r_1} &= \frac{2^{r_1}}{n} \binom{n}{r_0, r_1, r_2} \sum_{i=0}^{n-2} \binom{n-2}{i} \\
&= \frac{2^{r_1}}{n} \binom{n}{r_0, r_1, r_2} 2^{n-2} \\
&= \frac{2^{n+r_1-2}}{n} \binom{n}{r_0, r_1, r_2}.
\end{aligned}$$

The second last equality follows by binomial theorem. □

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this work, plane Husimi graphs have been enumerated according to outdegree sequence (Theorem 3.1.2), root degree (Lemma 3.1.5) and the number of vertices with a given outdegree (Theorem 3.1.7). We have also enumerated bicoloured plane tree-like structures with root of degree 1 in Proposition 3.2.1 and of root degree 2 in Proposition 3.2.6. Finally, we have also introduced and enumerated d-ary Husimi graphs according to block types in Theorem 4.1.1, number of blocks (Corollary 4.1.3), number of leaves (Theorem 4.2.1), for which an equivalent result for plane Husimi graphs was obtained earlier by Okoth in [15]. Outdegree sequence was also used as a parameter to enumerate d-ary Husimi graphs. The result is obtained in Theorem 4.3.1.

5.2 Recommendations

There are a number of ways in which this work can be extended. d-ary Husimi graphs have not been enumerated according to degree of the root. It is therefore recommended that a closed formula be obtained for the number of such graphs given the root degree. Eu, Seo and Shin [7] enumerated plane trees according to first children, non-first children and level. It would be therefore interesting to obtain equivalent results for plane Husimi graphs and d-ary Husimi graphs. Lastly, it is recommended that future work may involve obtaining the number of coloured plane Husimi graphs if more than two colours are used. The same can be extended to d-ary Husimi graphs. Forests of plane and d-ary tree-like structures can also be enumerated.

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