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RESEARCH ARTICLE

Gravitation in Flat Euclidean Spacetime Frame: Unified Electrogravity and Magnetogravity Forces

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ABSTRACT

An effective description of physics requires an appropriate geometrical frame. Three-dimensional Euclidean space provides the geometrical frame for non-relativistic physics. A derivation of an imaginary temporal axis $-ic\hat{q}$ the speed, $\hat{q}$ the unit wave-vector of light, extends the standard Euclidean space into a well-defined four-dimensional Euclidean spacetime frame, which provides the natural mathematical framework for relativistic physics. The basic elements of the Euclidean spacetime frame are fully specified four-component complex vectors satisfying standard vector operations and vector identities. In developing a theory of gravitation in the Euclidean *spacetime* frame, we have used the Lense-Thirring spacetime metric of linearized general relativity to derive an appropriate complex four-component gravitational field potential vector. Taking the curl of the field potential vector provides a unified complex gravitational field strength composed of electric-type and magnetic-type components. Taking the cross-product of the complex four-component velocity and the field strength provides a unified complex gravitational force intensity composed of gravitoelectric and gravitomagnetic components. Application to the motion of a gyroscope in the gravitational field of the earth provides the standard results of frame-dragging and geodetic effects as determined in linearized general relativity theory.

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1. Introduction

So far, the four-dimensional Minkowski spacetime frame has provided the geometrical framework for describing the more accurate relativistic dynamics. But a serious, though unnoticed, drawback of the Minkowski spacetime frame is that it does not specify a unit vector in the temporal direction, yet all the three associated spatial coordinate axes, namely the x-axis, y-axis and z-axis are specified by their respective unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. This means that in the Minkowski space-time frame, the four-component vectors, i.e., four-vectors, can only be expressed in terms of their components in contravariant or covariant forms, but not in the standard form for representing the three-component vectors in the three-dimensional Euclidean space frame. Mathematical operations with four-vectors in the Minkowski spacetime frame do not follow exactly the same procedure as the standard mathematical operations with the three-component vectors in the corresponding three-dimensional Euclidean space. In particular products or partial derivatives such as divergence and curl of four-vectors are conveniently evaluated if expressed in contravariant and covariant form. Even though the dot-product and the divergence of Minkowski four-vectors can be obtained in the standard Euclidean form if the temporal component is defined to be pure imaginary, the cross-product and the curl of the four-vectors with real or pure imaginary temporal components cannot be obtained in the standard Euclidean form, but can only be evaluated in terms of their generalized components expressed in appropriate tensor form. This leads to the inescapable conclusion that the specification of the four-dimensional Minkowski geometrical framework with real or imaginary temporal axis as an extension of the three-dimensional

Euclidean space is not complete. This is essentially due to the fact that the Minkowski framework does not explicitly specify the temporal axis unit vector, even though the mathematical operations imply that the temporal axis is orthogonal to all three spatial coordinate axes.

In recent work [1], [2], the present author derived the unit vector along the temporal axis and identified it as the unit wave vector $\hat{q}$ in the direction of propagation of light. It emerges that the temporal axis is an imaginary axis specified by the unit wave vector of light. This has provided a fully specified four-dimensional space-time geometrical frame with an imaginary temporal axis which can be interpreted as a mathematically consistent extension of standard three-dimensional Euclidean space to a four-dimensional Euclidean space-time frame. Four vectors with pure imaginary components and the related mathematical operations in the four-dimensional Euclidean *spacetime* frame now take exactly the same form as in the standard three-dimensional Euclidean space. Details of derivations, four-vector definitions, mathematical operations and four-vector identities in the four-dimensional Euclidean spacetime frame (complex spacetime frame) are presented in [1], [2]. We observe that an imaginary time *it* (imaginary temporal coordinate *ict*) was first introduced in four-dimensional spacetime frame by Minkowski [3].

2. Euclidean Spacetime Frame

In the specification of the Euclidean *spacetime* frame, the temporal unit vector $\hat{q}$ is taken to be of general orientation relative to the mutually orthogonal spatial axes unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ according to the general orientation conditions:

$$\hat{\imath}\cdot\hat{\jmath}=0;\hat{\jmath}\cdot\hat{k}=0;\hat{k}\cdot\hat{\imath}=0;\hat{q}\cdot\hat{\imath}\neq 0;\hat{q}\cdot\hat{\jmath}\neq 0;\hat{q}\cdot\hat{k}\neq 0 \tag{1}$$

The Euclidean spacetime frame is characterized by an imaginary temporal axis specified by the unit wave vector $\hat{q}$ of light propagation. We shall call a general complex four-component vector defined within the Euclidean *spacetime* frame, specified by all four spacetime unit vectors $\hat{q},\ \hat{\imath},\ \hat{\jmath},\ \hat{k}$, a *Euclidean four-vector*. We call the complex four-component gradient vector defined within the Euclidean spacetime frame the *Euclidean four-gradient*. We denote a Euclidean four-vector by an upper-case letter symbol such as V while the usual three-component vector in standard three-dimensional Euclidean space is denoted by a bold-faced letter symbol such as **V** or a conventional symbol with an over-arrow, such as $\vec{O}$, where necessary. The Euclidean four-gradient is denoted by ∇ while the usual three-component gradient vector in three-dimensional Euclidean space is denoted by an over-arrow in the form $\vec{\nabla}$.

A general Euclidean four-vector U, event four-vector X, displacement four-vector dX, velocity four-vector V and gradient four-vector ∇ are defined in the form:

$$U=-ic\varnothing\hat{q}+\vec{U};\ X=-ict\hat{q}+\vec{r};\ dX=-icdt\hat{q}+d\vec{r};\ V=-ic\hat{q}+\vec{v};\ \nabla=\frac{i}{c}\frac{\partial}{\partial t}\hat{q}+\vec{\nabla} \tag{2}$$

where c is the speed of light, $\vec{r}$ the position vector and $\vec{v}$ the velocity defined as usual in three-dimensional Euclidean space. With Euclidean four-vectors taking the standard vector form with all unit vectors specified according to (2), mathematical operations within the four-dimensional Euclidean spacetime frame follow exactly the same procedure as the well-established mathematical operations with three-component vectors in the standard three-dimensional Euclidean space.

The basic mathematical operations such as addition and subtraction, dot and cross products, divergence and curl of Euclidean four-vectors, and the Euclidean gradient of a scalar, are defined and evaluated in the standard vector operation forms:

$$W=U\pm Q=-ic\left(U_q\pm Q_q\right)\hat{q}+\left(\vec{U}\pm\vec{Q}\right);\ U\cdot Q=\vec{U}\cdot\vec{Q}-c^2U_qQ_q-ic\hat{q}\cdot\left(U_q\vec{Q}+\vec{U}Q_q\right)$$

$$U\times Q=\vec{U}\times\vec{Q}-ic\hat{q}\times\left(U_q\vec{Q}-\vec{U}Q_q\right);\ \nabla\times Q=\vec{\nabla}\times\vec{Q}-i\left(\frac{1}{c}\frac{\partial\vec{Q}}{\partial t}+\vec{\nabla}\left(cQ_q\right)\right)\times\hat{q}$$

$$\nabla\cdot Q=\frac{\partial Q_q}{\partial t}+\vec{\nabla}\cdot\vec{Q}+i\hat{q}\cdot\left(\frac{1}{c}\frac{\partial\vec{Q}}{\partial t}-\vec{\nabla}\left(cQ_q\right)\right);\ \nabla\varnothing=\frac{i}{c}\frac{\partial\varnothing}{\partial t}\hat{q}+\vec{\nabla}\varnothing \tag{3}$$

In detailed calculations presented in [1], [2], we have proved that Euclidean four-vectors satisfy exactly all forms of standard vector identities for three-component vectors in three-dimensional Euclidean space. For completeness in defining the mathematical operations in the Euclidean spacetime frame, we present the final results of the basic Euclidean four-vector identities, which for a scalar function $\varnothing$ and Euclidean four-vectors $U,\ Q,\ W$ have been obtained as:

$$\nabla \times \nabla\varnothing = 0;\ \nabla \cdot (\nabla \times U) = 0;\ \nabla \times (\nabla \times U) = \nabla (\nabla \cdot U) - \nabla^2 U \tag{4a}$$

$$\nabla \cdot (\varnothing U) = \nabla\varnothing \cdot U + \varnothing\nabla \cdot U;\ \nabla \times (\varnothing\nabla) = \nabla\varnothing \times U + \varnothing\nabla \times U;\ \nabla \cdot (U \times Q) = Q \cdot (\nabla \times U) - U \cdot (\nabla \times Q) \tag{4b}$$

$$\nabla (U \cdot Q) = U \times (\nabla \times Q) + Q \times (\nabla \times U) + (U \cdot \nabla)\, Q + (Q \cdot \nabla)\, U \tag{4c}$$

$$\nabla \times (U \times Q) = U (\nabla \cdot Q) - Q (\nabla \cdot U) - (Q \cdot \nabla)\, U - (U \cdot \nabla)\, Q \tag{4d}$$

$$W \times (U \times Q) = U (W \cdot Q) - Q (W \cdot U);\, W \times (U \times Q) + U \times (Q \times W) + Q (W \times U) = 0 \tag{4e}$$

$$U \cdot (U \times Q) = 0;\ U \cdot (Q \times W) = Q \cdot (W \times U) = W \cdot (U \times Q) \tag{4f}$$

The basic algebraic operations presented in (4)–(10) constitute the mathematical foundation for formulating theories of dynamics of physical systems such as atoms in Euclidean space-time frames.

3. Physics in Euclidean Spacetime Frame

In a systematic formulation of physics in the Euclidean spacetime frame, we apply the mathematical property that the basic elements of the Euclidean spacetime frame are Euclidean four-vectors, which as defined above, are complex four-component vectors. The basic dynamical elements, the mechanical elements and the electromagnetic elements, of a physical system are defined as Euclidean four-vectors. As usual, the fundamental physical properties of matter are mass and electric charge, which generate appropriately defined force fields. We interpret a force field, with the generating mass or electric charge at the center (origin), as a bounded four-dimensional Euclidean spacetime frame. A mechanical (mass-generated) or electromagnetic (electric charge-generated) force field is characterized by a Euclidean field potential four-vector. In general, mathematical operations with the physical Euclidean four-vectors have algebraic properties which reveal fundamental features of dynamics within a Euclidean spacetime frame.

Consistently with standard descriptions of physics in various geometrical frames, we identify the basic dynamical properties of physics in the Euclidean *spacetime* frame as mass m and electric charge q. These basic dynamical properties are defined as the temporal components of the respective Euclidean linear momentum and electric current density four-vectors P, J defined by

$$P = mV = -imc\hat{q} + \vec{p};\ J = -ic\rho\hat{q} + \vec{J};\ \vec{p} = m\vec{v};\ \vec{v} = \frac{d\vec{r}}{dt} \tag{5}$$

where $\vec{v}$ is the velocity, $\vec{p}$ linear momentum, ρ electric charge density and $\vec{J}$ electric current density, all defined as usual in three-dimensional Euclidean space. The Euclidean linear momentum four-vector is equivalently interpreted as energy-momentum four-vector defined by introducing relativistic energy $E = mc^2$ in the form [4], [5].

$$P = -i\frac{E}{c}\hat{q} + \vec{p};\ E = mc^2 \tag{6}$$

Other physical quantities such as orbital and spin angular momenta, together with the associated orbital and spin magnetic moments, etc., can be defined as desired.

In general, physics in the Euclidean spacetime frame naturally satisfies the basic conservation laws and invariance under Lorentz transformation. Application of the conservation laws and invariance under the Lorentz transformation easily provide the fundamental relativistic properties of time dilation and mass increase with speed, which have been obtained together with the corresponding *spacetime* metric ds and conserved energy E in the form [1], [2].

$$dt = \eta^{-\frac{1}{2}}\gamma d\tau;\ ds = \eta^{\frac{1}{2}}\gamma^{-1}cdt;\ m = \eta^{-\frac{1}{2}}\gamma m_0;\ E^2 = p^2c^2 + \eta^{-1}{m_0}^2c^4 \tag{7}$$

where τ is the proper time, m_0 the rest mass, γ the usual Lorentz transformation factor and η a Euclidean spacetime frame modification factor obtained in the form

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}};\ \eta = \sqrt{1+\frac{4\left(\hat{q}.\vec{v}\right)^2}{c^2\gamma^{-2}}} \tag{8}$$

In the special case where the temporal unit vector $\hat{q}$ is orthogonal to all the three spatial unit vectors $\hat{i}, \hat{j}, \hat{k}$, giving $\hat{q}.\vec{p} = 0$, $\eta = 1$, the fundamental relativistic properties in (7) reduce to the standard forms within the *Minkowski spacetime* frame associated with Einstein's special theory of relativity.

In the Euclidean spacetime frame in our case the gravitational field, a force field is characterized by a Euclidean field potential four-vector A and defined by

$$A = -ic\varnothing\hat{q} + \vec{A} \tag{9}$$

where $c\varnothing$ is the scalar and $\vec{A}$ is the vector potential. The field strength F is obtained as the curl of the field potential four-vector A and the force intensity F_i is obtained as the cross-product of the velocity four-vector V and the field strength in the form

$$F = \nabla \times A;\ F_i = \frac{1}{c} V \times F \tag{10}$$

The Euclidean field potential four-vector, with the associated field strength and force intensity in (9), (10) are defined in general form, which can be applied to any force field such as electromagnetic and gravitational fields in a flat spacetime frame; only the dynamical properties of the field are specified in each case. We now develop an application to gravitation in a flat Euclidean spacetime frame in the next section.

4. The Euclidean Gravitational Field

An elegant property which has emerged is that the mathematical operations with Euclidean four-vectors provide a natural transition from non-relativistic physics in three-dimensional Euclidean space to relativistic physics in four-dimensional Euclidean spacetime frame. This property means that the Euclidean *spacetime* frame may be considered as the natural geometrical framework for developing the basic relativistic theory of gravitation, being a fundamental example of a general Euclidean mechanical force field.

We now introduce the Euclidean gravitational force field. To determine the appropriate form of the gravitational Euclidean field potential four-vector, we begin by observing that in Einstein's general theory of relativity, the gravitational field potential is defined by a rank-2 symmetric metric tensor in Riemann geometry. The general form of the metric tensor characterizes gravitation in a non-inertial spacetime frame. Noting that in a weak gravitational field, linearization of Einstein's general theory of relativity reduces to *gravitoelectromagnetism* (GEM), which governs relativistic gravitation in an inertial flat four-dimensional Minkowski spacetime frame [6]–[12], we use the standard Lense-Thirring spacetime metric in GEM to derive the appropriate form of the (weak) gravitational field potential four-vector. We developed the procedure for determining the GEM field potential four-vector within the flat Minkowski spacetime frame in [13], which we review briefly.

In GEM, the standard Lense-Thirring spacetime metric is obtained in the form [6]–[11], [13].

$$ds^2 = (1+2\varnothing)\,d\tau^2 + 2\vec{h}\cdot d\vec{r}d\tau - (1-2\varnothing)\,d\vec{r}^2 \tag{11a}$$

which we reorganize to bring the terms involving $d\tau$ together as

$$ds^2 = \left\{(1+2\varnothing) + 2\vec{h}\cdot\vec{v}\right\}d\tau^2 - (1-2\varnothing)\,d\vec{r}^2;\ \vec{v} = \frac{d\vec{r}}{d\tau} \tag{11b}$$

where τ is the proper time and $\vec{v}$ is the velocity of a moving mass, such as a gyroscope in the GEM field. Introducing the speed of light c as appropriate in the temporal component of (11b) gives the form

$$ds^2 = \left\{\frac{1}{c}(1+2\varnothing)\,c + 2\vec{h}\cdot\vec{v}\right\}d\tau^2 - (1-2\varnothing)\,d\vec{r}^2 \tag{11c}$$

where we now recognize the GEM field scalar and vector components $\frac{1}{c}(1+2\varnothing)$, $2\vec{h}$ together with the corresponding velocity four-vector components c, $\vec{v}$. This allows us to express the coefficient of the temporal component in (11c) in standard spacetime covariant form

$$ds^2 = A_g^{\mu} V_{\mu} d\tau^2 - (1-2\varnothing)\,d\vec{r}^2;\ \mu = 0, 1, 2, 3 \tag{11d}$$

where we identify A_g^μ as the GEM field potential four-vector and V_μ as the velocity four-vector of the moving mass obtained in respective forms

$$A_g^\mu = \left(\varnothing_g, \vec{A}_g\right); \ \varnothing_g = \frac{1}{c}\left(1+2\varnothing\right); \ A_g = 2\vec{h}; \ V_\mu = \left(c, -\vec{v}\right) \tag{11e}$$

We have thus used the Lense-Thirring spacetime metric to determine the appropriate form of the GEM field potential four-vector A_g^μ and the corresponding form of the velocity four-vector V_μ of a mass moving in the GEM field.

The GEM field potential four-vector $A_g^\mu = \left(\varnothing_g, \vec{A}_g\right)$ determined from the Lense-Thirring spacetime metric of linearised general relativity characterizes a weak gravitational force field in a flat four-dimensional Minkowski spacetime frame. In the completely defined Euclidean spacetime frame where the imaginary temporal axis is specified by a unit vector, the GEM field potential A_g^μ is now defined as the Euclidean gravitational field potential four-vector A_g in the form

$$A_g = -ic\varnothing_g\hat{q} + \vec{A}_g; \ \varnothing_g = \frac{1}{c}\left(1+2\varnothing\right); \ \vec{A}_g = 2\vec{h} \tag{12}$$

With the Euclidean gravitational field potential four-vector determined as in (12), the Euclidean gravitational field strength, force and related dynamical quantities in relativistic gravitation can now be obtained.

The gravitational field strength F is obtained as the curl of the gravitational field potential four-vector A_g according to (10), which on applying the general form of the curl in (3), takes the form

$$F = \vec{B}_g + i\vec{E}_g \times \hat{q}; \ \vec{B}_g = \vec{\nabla} \times \vec{A}_g; \ \vec{E}_g = -\vec{\nabla}\left(c\varnothing_g\right) - \frac{1}{c}\frac{\partial \vec{A}_g}{\partial t} \tag{13}$$

where, by definition, the field strength component $\vec{B}_g$ generates deflection, while $\vec{E}_g$ generates translational motion in the Euclidean gravitational force field. Comparison with standard electromagnetic field strength leads to the interpretation that the component $\vec{B}_g$ generates magnetic-type effects, while the component $\vec{E}_g$ generates electric-type effects in the gravitational force field. Consequently, we identify the component $\vec{E}_g$ as the *gravitomagnetic* field strength and the component $\vec{E}_g$ as the *gravitoelectric* field strength. In the non-relativistic limit, the gravitoelectric field strength $\vec{E}_g$ reduces to the familiar Newton's gravity field strength or gravitational acceleration $\vec{g}$. The important dynamical property emerging in (13) is that the Euclidean gravitation field strength F is a unified *electromagnetic*-type field strength, which is generally called *gravitoelectromagnetic* (GEM) field strength in linearized general relativity theory [6]–[12].

The gravitational force intensity F_i in the Euclidean gravitational force field is obtained as the cross-product of the Euclidean density four-vector V and the field strength F according to (10). Substituting V from (2) and F from (13) into the definition of F_i in (10), then applying the general form of the cross-product from (3) provides the final form:

$$F_i = F_L + iF_O; \ F_L = \vec{E}_g + \frac{\vec{v}}{c} \times \vec{B}_g - \left(\hat{q}.\vec{E}_g\right)\hat{q}; \ F_O = \vec{B}_g \times \hat{q} + \frac{\vec{v}}{c} \times \left(\vec{E}_g \times \hat{q}\right) \tag{14}$$

We identify F_L as a Lorentz-type (electric-type) force intensity and F_O as an orbital magnetic-type force intensity. The gravitational force intensity F_i is therefore a unified force composed of a Lorentz electric-type component F_L which we define as a *gravitoelectric* force intensity and an orbital magnetic-type component F_O which we define as a *gravitomagnetic* force intensity.

Noting that $\vec{B}_g \times \hat{q}$ is normal to the plane between $\vec{B}_g$ and $\hat{q}$, and that $\vec{E}_g \times \hat{q}$ is normal to the plane between $\vec{E}_g$ and $\hat{q}$, we redefine the gravitomagnetic force intensity F_O in the familiar form:

$$F_O = \vec{B}_g \uparrow + \frac{\vec{v}}{c} \times \vec{E}_g \uparrow; \ \vec{B}_g \uparrow = \vec{B}_g \times \hat{q}; \ \vec{E}_g \uparrow = \vec{E}_g \times \hat{q} \tag{15}$$

We observe that, apart from a missing factor $\frac{3}{2}$ in the second component, the gravitomagnetic force intensity obtained here in (18) or (19) agrees with the form obtained in the calculations of frame-dragging and geodetic effects using the geodesic equation for the spin four-vector of a gyroscope in linearized general relativity theory [7], [8]. We also notice that the Lorentz-type force has a component that is induced by the motion and another term directed in the temporal axis as additional terms. Force is the rate of change of momentum, so we can write

$$\frac{d\vec{P}}{dt} = m\vec{E}_g + \frac{\vec{v}}{c} \times m\vec{B}_g - m\left(\hat{q}.\vec{E}_g\right)\hat{q} \tag{16}$$

The other component can be interpreted as the rate of change of spin

$$\frac{dm\vec{S}}{dt} = m\vec{B}_g \times \hat{q} + \frac{\vec{v}}{c} \times \left(m\vec{E}_g \times \hat{q}\right) \tag{17}$$

It is the magnetic-type forces that cause a change in the spin of a gyroscope moving in the gravitational field of the earth. Relativistic effects are therefore obtained by taking the cross-product of the spin-vector and the magnetic-type forces.

$$\frac{gm'}{2mc}\vec{S} \times \left(\vec{B}_g \times \hat{q} + \frac{\vec{v}}{c} \times \left(\vec{E}_g \times \hat{q}\right)\right);\ g = 2;\ m' = m \tag{18}$$

Finally, we have the frame-dragging and geodetic effects

$$\vec{\Omega}_{FD} = -\frac{3}{2}m\vec{B}_g \uparrow \times \vec{S};\ \vec{\Omega}_{GE} = -\frac{3}{2}\frac{\vec{v}}{c} \times m\vec{E}_g \uparrow \times \vec{S} \tag{19}$$

5. Applications

From Astronomical point of view, reference frames serve as the observational perspectives from which we perceive the motion of objects in spacetime. When a massive object, such as a rotating black hole, drags spacetime along with its rotation, the effect is observable from different reference frames. From the perspective of an observer on the site to the rotating mass, the dragging of spacetime appears minimal or non-existent. However, for an observer distant from the rotating mass, the frame-dragging effect becomes more pronounced, influencing the motion of nearby objects and altering their trajectories. Therefore, reference frames provide the framework for understanding relativistic effects that depend on the observer's motion relative to the rotating mass [14].

Similarly, geodetic effects are intimately connected to reference frames in general relativity. These effects demonstrate the idea of curvature of *spacetime* caused by massive objects. It is the point to consider when explaining the fact that objects follow geodesic paths. The choice of reference frame affects how these geodesic paths are perceived. In a freely falling reference frame, where an observer experiences no gravitational forces, objects follow geodesic paths dictated by the curvature of *spacetime*. However, in a non-inertial reference frame, such as one attached to a massive body like the Earth, the apparent motion of objects is influenced by additional forces, such as gravitational acceleration, which can obscure the underlying geodetic effects. Therefore, reference frames are essential for disentangling the effects of curvature from other forces and understanding the true nature of geodetic motion in curved spacetime. By studying these effects, we can validate and refine our understanding of general relativity, which serves as the cornerstone of modern theoretical physics [15].

In astrophysics, frame dragging influences the behaviour of massive rotating objects such as black holes and neutron stars. This effect affects the orbits of nearby objects and the emission of gravitational waves, providing valuable insights into the dynamics of these systems [16]. Observations of frame-dragging contribute to our understanding of astrophysical phenomena and help confirm the predictions of general relativity in extreme gravitational environments. For example, the detection of gravitational waves from merging black holes provides direct evidence of frame dragging effects and offers insights into the properties of these enigmatic objects [17].

Understanding relativistic effects is essential for precise navigation systems, particularly in space missions where accurate positioning is critical. Corrections based on general relativity, including geodetic effects, are necessary for achieving the high levels of accuracy required for GPS applications on Earth and in space. For instance, the Global Positioning System relies on corrections derived from general relativity to ensure accurate positioning and timing information for navigation purposes.

6. Conclusion

As it is, we have demonstrated that it is possible to handle *gravitoelectromagnetism* using vector mathematics. It is quite clear now that the motion of a body in the gravitational field induces an additional component to the component that was identified by Newton. The practical implications of relativistic effects extend beyond theoretical physics, influencing our understanding of celestial mechanics, the behaviour of compact astrophysical objects, and even the design of space missions. As we continue to push the boundaries of our understanding, these phenomena will undoubtedly remain at the forefront of gravitational research, guiding us toward new frontiers in both theoretical and observational exploration.

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Conflict of Interest

Authors declare that they do not have any conflict of interest.

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