NUMERICAL SOLUTION OF KORTEWEG-DE VRIES EQUATION

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ABSTRACT

The Korteweg-de Vries (KdV)is a mathematical model of waves on shallow water surfaces. The mathematical theory behind the KdV equation is rich and interesting, and, in the broad sense, is a topic of active mathematical research.The equation is named after Diederik Korteweg and Gustav de Vries.

It has long been known that conservative discretization schemes for the KdV and other nonlinear equations tend to become numerically unstable. Although finite difference approximations have been used, there are always instabilities of the solutions obtained.

In this work we solved the Korteweg-de Vries (KdV) equation using an explicit finite difference method, subject to various boundary conditions which are travelling wave solutions to the KdV equation. The methodology involved carefully designing conservative finite difference discretization that can remain stable and deliver sharp solution profiles fora long time. We then determined the accuracy of the finite difference scheme by comparing the graphical outputs of the numerical results.

Chapter 1

INTRODUCTION

1.1 Introduction

The Korteweg-de Vries (KdV)equation is given by:

$$u_t + uu_x + u_{xxx} = 0 \tag{1.1}$$

Equation (1.1) describes one-dimensional shallow-water waves, with small but finite amplitudes. Form the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasma to shock waves, from tornados to the great red spot of Jupiter, from elementary particles of matter to the elementary particles of thought.

More recently, the KdV equation has been found to describe wave phenomena in plasma physics [3,23], anharmonic crystals [12,24] and bubble-liquid mixtures [20,21]. The KdV equation is also relevant to the discussion of the interaction between nonlinearity and dispersion, just as the well-known Burgers equation shows the features of the interaction between nonlinearity and dissipation.

The Korteweg-de Vries Equation (1.1) is nonlinear because of the second summand and is of third order because of the third derivative as highest in the third summand.

For appropriate initial functions Lax and Sjöbrerg[13,17] have shown the existence and uniqueness of solutions of the KdV equation. Approximate solutions in the form of expansions were given by Broer [2], while Hoogstraten [8] obtained asymptotic solutions for slowly varying wave trains.

Gardner, Lax et al. [6,14] described analytic considerations concerning the existence of solitary waves in solutions of certain initial-value problems. Zabusky and Kruskal [25]encountered this appearance of solitary waves in studying the results of a numerical analysis.

Jainp, Shankar and Bhardwajs [9]in 1996 developed an algorithm by using splitting and quintic spline approximation to solve the KdV equation.

Moreover, it is well known that unexpected instabilities occasionally arise for reasonable-looking finite difference discretizations.

1.2 Literature Review

The Korteweg-de Vries(KdV) equation(1.1) was first derived in 1895 by Korteweg and de Vries to model water waves in a shallow canal. Their goal was to settle a long-standing question: whether a solitary wave could persist under these conditions. Based on his personal observations of such waves since 1830's, the natural John Scott Russell insisted that such do occur, but several prominent mathematicians, including Stokes, were convinced they were impossible.

Korteweg and de Vries proved Russell was correct by finding explicit closed-form, traveling-wave solutions to their equation that more over decay rapidly and so represent a highly localized moving hump. The Kdv eqation did not receive significant further attention until 1965, when Zabusky and Kruskal [25] published the results of their numerical experimentation with the equation. Their computer generated approximate solutions to the KdV equation indicating that any localized initial profile that was allowed to evolve according to the KdV equation eventually consisted of a finite set of localized traveling waves of the same shape as the original solitary waves discovered in 1895. Further more, when two localized disturbances collided, they would emerge from the collision again as another pair of traveling waves with a shift in phase as the only consequence of their interaction. Since the "solitary waves" which made up these solutions seemed to behave like particles, Zabusky and Kruskal coined the name "soliton" to describe them.

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self sufficient entity, a "thing" displaying many properties of a particle.

The inverse scattering theory [15-17] provides analytic solutions in principle. This method has had enormous impact on the analysis of the KdV equation and other completely integrable equations,but can also be used numerically.

Helge Holden, Kenneth Hvistendahl, Karlsen and Nils Risebro [16] applied the method of operator splitting on the generalized KdV equation

$$u_t + f(u)_x + \varepsilon u_{xxx} = 0 \tag{1.2}$$

by solving the nonlinear conservation law

$$u_t + f(u)_x = 0 \tag{1.3}$$

and the linear dispersive equation

$$u_t + \varepsilon u_{xxx} = 0 \tag{1.4}$$

sequentially. They proved that if the approximation obtained by operator splitting converges, then the limit function is a weak solution of the generalized KdV equation. A.C.Vliegenthart[17] used a centred finite difference scheme for the KdV equation which also has been used by Zabusky and Krusal[13] which is given by

$$u_j^{n+1} = u_j^{n-1} - \frac{1}{3}\varepsilon\frac{\Delta t}{\Delta s}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-i}^n) - \mu\frac{\Delta t}{(\Delta s)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n) \tag{1.5}$$

In 1996 Jain, Shankar and Bhardwaj [9]solved the KdV equation numerically by using splitting method and quintic approximation technique.

It has long been known that conservative discretization schemes for the KdV equation and other nonlinear equations tend to become numerically unstable.

To date, a solution of equation (1.1) by carefully designing an explicit finite difference discretization that remains stable for a long time, is lacking.

1.3 Statement of the Problem

It has long been known that conservative discretization schemes for the KdV equation and other nonlinear equations tend to become numerically unstable. Although finite difference approximations have been used, there are always instabilities of the solutions obtained.

There is therefore need to develop a scheme which can solve the Korteweg-de Vries equation using an explicit finite difference numerical scheme which gives a stable solution subject to variuos boundary conditions, always with consistent initial conditions which are traveling wave solutions to equation (1.1)

1.4 Objective of the Study

The objectives of this study are:

- To find the solution of the KdV equation using explicit finite difference approximations.
- To analyze the stability of the finite difference discretization used
- To compare the graphical outputs of the numerical results to determine the accuracy of the finite difference scheme.
- To determine the parameters that give a stable result for the explicit finite difference scheme.

1.5 Significance of the Study

The results of this study provide an alternate numerical approach for solving the KdV equation with a finite difference scheme, which may be exploited to obtain a meaningful result. This is also a significant contribution to knowledge and further research.

1.6 Research Methdology

We have solved the KdV equation using an explicit numerical scheme subject to various boundary conditions, with consistent initial conditions which are traveling wave solutions to equation (1.1).

We introduced a finite difference scheme and determined its stability conditions. We also compared the graphical outputs of the numerical results to determine the accuracy of the finite difference scheme.

Chapter 2

BASIC CONCEPTS

2.1 Hyperbolic Partial Differential Equations

Hyperbolic (and parabolic) equations result from problems involving time as one independent variable and semi-infinite domain (time is unbounded). They require boundary and initial conditions. Hyperbolic equations generally originate from vibration problems or from problems where discontinuities can persist in time (shock waves).

Analytic solutions of two independent variables often use the method of characteristics, which reduce the solution to solving ordinary differential equations. Unlike elliptic and parabolic solutions, the influence of the domain on a particular point is limited in extent (see the following example).

Example (2.1): Solve $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$.

1. b.c(1):$u(x, 0) = f(x)$
2. b.c (2):$u_t(x, 0) = g(x)$

where b.c is short for boundary condition.

Use the alternative form $u_{\xi\eta} = 0$, where $\xi = x - ct$, and $\eta = x + ct$

Integrate $u(\xi, \eta) = F_1(\eta) + F_2(\xi)$ or $u(x,t) = F_1(x+ct) + F_2(x-ct)$

1. b.c. (1):$f(x) = F_1(x) + F_2(x)$

2. b.c. (2):$g(x) = cF_1'(x) - cF_2'(x)$

$u(x,t) = \frac{1}{2}[f(x+ct) - f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$ Note solution at (x_0, t_0) only depends on initial data: $x_0 - t_0 \leq x \leq x_0 + t_0$ This is the behaviour characteristic of all hyperbolic equations.

2.2 Difference Approximations To Derivatives

From the Taylor Series if $f(x)$ is a function of a variable x with h, a small change in x,then

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}(x) + 0(h^4)$

we have

$$u(x+h) = u(x) + hu^1(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + 0(h^4) \qquad (2.1)$$

and

$$u(x-h) = u(x) - hu^1(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + 0(h^4) \qquad (2.2)$$

If we add equation (2.1) to equation (2.2), we obtain

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + 0(h^4) \tag{2.3}$$

By solving equation (2.3) we get

$$u''(x) = (\frac{d^2u}{dx^2})_{x=x} = \frac{1}{h^2}\{u(x+h) - 2u(x) + u(x-h)\} + 0(h^2) \tag{2.4}$$

Further subtracting equation (2.2) from equation (2.1) yields

$$u(x+h) - u(x-h) = 2h'(x) + 0(h^2) \tag{2.5}$$

From which a solution for $u'(x)$ will be

$$u'(x) = \frac{1}{2h}\{u(x+h) - u(x-h)\} + 0(h^2) \tag{2.6}$$

Equations (2.5) and (2.6) represent central-difference approximations of the derivatives, and both have errors of $0(h^2)$ and can be approximated by the slope ofthe tangent at P in figure (2.1) by the slope of chord AB.

But solving equations (2.1) and (2.2) for $u'(x)$ directly yields

$$u'(x) = \frac{1}{h}\{u(x+h) - u(x)\} + 0(h) \tag{2.7}$$

representing the forward difference approximation which can also be given by slope of chord PB.

And

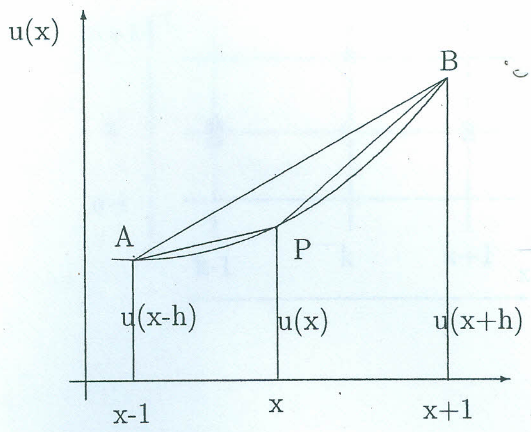

Figure 2.1: Approximations to derivatives

$$u'(x) = \frac{1}{h}\{u(x) - u(x-h)\} + 0(h) \quad (2.8)$$

representing the backward difference approximation also given by the slope of chord AP.

Note: The forward and backward difference approximations have error of order $0(h)$, where as the central difference approximations have error of order $0(h^2)$.

2.3 Finite-Difference Method

The finite-difference method replaces the continuous problem by a discrete or finite difference mesh or grid.

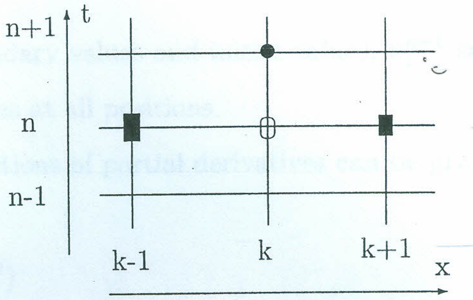

Figure 2.2: The Explicit finite-difference molecule

We define $x_i \equiv i\Delta x, x_{i+1} = (i+1)\Delta x$ for $u_i \equiv u(x_i)$

In a similar manner, the time domain is discretized to get:

$t^n \equiv n\Delta t$

For $u(t,x) : u_i^n \equiv u(n\Delta t, x_i)$

Finite-difference method also replaces the derivatives in a partial differential equation (PDE) with finite approximations.

Example (2.3.1)

With $u(t,x)$

The forward approximation is $(\frac{\partial u}{\partial t})_{t=t_x^n=x_i} \approx \frac{u_i^{n+1}-u_i^n}{\Delta t}$

The central difference approximation is $(\frac{\partial^2 u}{\partial x^2})_{t=t_x^n=x_i} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$ Assuming all values at t^n are known, solve for the unknown u_i^{n+1} as shown in figure (2.2)

$u_i^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n$

where $r = \frac{\Delta t}{\Delta x^2}$

Therefore, given boundary values and initial values, u_i^{n+1} can be calculated for all future times at all positions.

In general the approximations of partial derivatives can be given as

$(\frac{\partial u}{\partial t})_i^n = \frac{u_i^{n+1}-u_i^n}{\delta t} + 0(\delta t)$

$(\frac{\partial u}{\partial x})_i^n = \frac{u_i^{n+1}-u_{i-1}^n}{2\delta x} + 0(\delta x^2)$

$(\frac{\partial^2 u}{\partial x^2})_i^n = \frac{u_i^{n+1}-2u_i^n+u_{i-1}^n}{\delta x^2} + 0(\delta x^2)$

$(\frac{\partial^3 u}{\partial x^3})_i^n = \frac{u_{i+2}^n-2u_i^{n+1}+2u_{i-1}^n+u_{i-2}^n}{\delta x^3} + 0(\delta x^3)$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

2.4 Stability of Solutions

Fourier or Von Neumann analysis shows that a difference expression is stable if $0 < r \leq \frac{1}{2}$ where $r = \frac{\Delta t}{\Delta x^2}$

In general, finite difference approximations of hyperbolic partial differential equations $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ has restrictions based on $c\frac{\Delta t}{\Delta x}$, which is the Courant-Friedrichs-Lewy (CFL) condition that states:

That a necessary condition for an explicit finite difference scheme to solve a hyperbolic (PDE) to be stable is that, for each mesh point, the domain of dependence of the finite difference approximation contains the domain of dependence of the PDE. That is if: $r \leq 1$, where $r = c\frac{\Delta t}{\Delta x}$, then the scheme is stable.

Chapter 3

WAVE AND KdV EQUATIONS

The wave equation and the Korteweg-de Vries equations are some of the hyperbolic partial differential equations which have practical applications in real life. We discuss them below. .

3.1 Wave Equation

Consider a simple example of a hyperbolic partial differential (or wave) equation with one spatial independent variable

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{3.1}$$

where c is the speed of the wave.

We take a rectangular net with constant intervals $h = \Delta x$, $k = \Delta t$. The equivalent finite difference approximation is given by:

$$U_{i,j+1} = r^2(U_{i-1,j} + U_{i+1,j}) + 2(1-r^2)U_{i,j} - U_{i.j-1} \tag{3.2}$$

where $x = i\Delta x, i = 1, 2, 3, ...n, t = j\Delta j, j = 1, 2, ...$

In the equation (3.2), we use the central difference formula for the derivatives with respect to t as well as with respect to x. If we let

$$r = k/h = (\frac{c\Delta t}{\Delta x}) \tag{3.3}$$

The first initial condition specifies $U_{i,0}$ on the line $t = 0$. We can use the second condition to find values on the line $t = k$ by using an "imaginary" boundary and the second order central difference formula

$$\frac{\partial U}{\partial t}|_{i,o} = \frac{U_{i,1-U_{i,-1}}}{2k} \tag{3.4}$$

Writing $g(i\Delta x) = g_i$, we have the approximation

$$U_{i,1} - U_{i,-1} = 2kg_i \tag{3.5}$$

that is, when $U_{i,-1}$ appears, we replace it by its value in equation (3.5), $U_{i,-1} = U_{i,1} - 2kg_i$. With $j = 0$ in equation (3.2) we have

$$U_{i,1} = r^2(U_{i-1,0} + U_{i+1,0} + 2(1 - r^2)U_{i,0} - U_{i,-1} \tag{3.6}$$

Upon replacing $U_{i,-1}$ with its value, from equation (3.5) and solving for $U_{i,1}$, we find

$$U_{i,1} = \frac{1}{2}r^2(f_{i-1} + f_{i+1}) + (1 - r^2)f_i + kg_i. \tag{3.7}$$

For the stability of the scheme let $z_{i,j} = u_{i,j} - U_{i,j}$ be the difference between the true solution at (i, j), $u_{i,j}$, and the finite difference solution

$U_{i,j}$. By Taylor's series, then we determine the truncation error for various terms and find the difference expression

$$Z_{i,j+1} = z_{i,j-1} + r^2(z_{i-1,j} + z_{i+1,j}) + 2(1-r^2)z_{i,j} + 0(k^4) + 0(k^2h^2) \quad (3.8)$$

Since U agrees with u on the initial line, $z_{i,0} = 0$, for all i, if we employ equation (3.2) in the first time step, and note that $z_{1,0} = 0$, we find

$$z_{i,1} = 0(k^3) \quad (3.9)$$

To investigate stepwise stability of equation (3.8), we examine the propagation effect of a single term of the form $exp[(-1)^{\frac{1}{2}}\beta x]$, where β is any real number, say, along the line $t = 0$. the errors are propagated according to the form of equation (3.8), that is, equation (3.2) with $U_{i,j}$ replaced by $z_{i,j}$. The initial condition, according to our methodology is

$$z_{i,0} = exp[(-1)^{\frac{1}{2}}\beta ih] \quad (3.10)$$

Upon attempting a solution by separation of variables, we try

$$z_{i,j} = exp(\alpha jk)exp[(-1)^{\frac{1}{2}}\beta ih] \quad (3.11)$$

Setting this into equation (3.8) results in $e^{\alpha k} + e^{-\alpha k} = (2 - 4r^2 sin^2\frac{1}{2}\beta h)$ which can be expressed as the quadratic equation

$$(e^{\alpha k})^2 - 2(1 - r^2 sin^2\frac{1}{2}\beta h)e^{\alpha k} + 1 = 0 \quad (3.12)$$

in $e^{\alpha k}$.

To avoid an increasing exponential solution as $j \to \infty$, it is necessary that $|e^{\alpha k}| \leq 1$ for all real values of β. From equation (3.12) the product of the two values of $e^{\alpha k}$ is clearly 1. Thus there exist solutions of the form of equation(3.11) that grow exponentially as j increases unless the discriminant of equation (3.12) is non-positive. That is

$$(1 - 2r^2 sin^2 \frac{1}{2}\beta h)^2 - 1 \leq 0 \tag{3.13}$$

Example 3.1

Solve the wave equation

$u_{tt} = u_{xx}$, $0 < x < 1$, $t \geq 0$

subject to the boundary conditions

$u(0, t) = 0 = u(1, t)$, $t \geq 0$

and initial conditions

$u(x, 0) = sin\pi x$, $0 < x < 1$, $u_i(x, 0) = 0$, $0 < x < 1$

Solution 3.1

The analytic solution is easily obtained as

$$u(x, t) = sin\pi x cos\pi t \tag{3.14}$$

Using the explicit finite difference scheme of equation (3.4) with $r = 1$, we obtain

$$u(i, j + 1) = u(i - 1, j) + u(i + 1, j) - u(i, j - 1), j \geq 1 \tag{3.15}$$

For $j = 0$, substituting

$u_t = \frac{u(i,1)-u(i,-1)}{2\Delta t} = 0$

or $u(i,1) = u(i,-1)$

into equation (3.15) gives the starting formula

$$u(i,1) = \frac{1}{2}[u(i-1,0) + u(i+1,0)] \tag{3.16}$$

Since $c = 1$ and $r = 1$, $\Delta t = \Delta x$. Also, since the problem is symmetric with respect to $x = 0.5$, we solve for u using equations (3.15) and (3.16) within $0 < x < 0.5$, $t \geq 0$. We can either calculate the values by hand or write a computer program. The results in table (3.1) is obtained for $\Delta t = \Delta x = 0.1$. The finite difference solution agrees with the exact solution in equation (3.14) to six decimal places. The accuracy of the finite difference solution can be increased by choosing a smaller spatial increment Δx and a smaller time increment Δt.

x	0.1	0.2	0.3	0.4	0.5	0.6
0.0	0.3090	0.5879	0.8990	0.9511	1.0	0.9511
0.1	0.2939	0.5590	0.7694	0.9045	0.9511	0.9045
0.2	0.2500	0.4755	0.6545	0.7694	0.8090	0.7694
0.3	0.1816	0.3455	0.4755	0.5590	0.5878	0.5590
0.4	0.0955	0.1816	0.2500	0.2939	0.3090	0.2930
0.5	0	0	0	0	0	0
0.6	-0.0955	-0.1816	-0.2500	-0.2939	-0.3090	-0.2939
0.7	-0.1816	-0.3455	-0.4755	-0.5590	-0.5878	-0.5590
↕	↕	↕	↕	↕	↕	↕

Table 3.1:Results for $\Delta x = \Delta t = 0.1$

3.2 Exact Solution To The KdV Equation

We recall that the simplest form $u(x,t) = f(x-ct)$ which is solution to the simple partial differential equation $u_t + cu_x = 0$ where c denotes the speed of the wave. For the well known wave equation $u_{tt} - c^2 u_{xx} = 0$, the famous d'Alembert solution leads to two wave fronts represented by the terms $f(x-ct)$ and $f(x+ct)$.

Hence we start with a trial solution

$$u(x,t) = z(x - \beta t) \equiv z(\xi) \tag{3.14}$$

where we denote the parameter c by β, the function f by z and the solution by ξ into equation (1.1) we are led to the Ordinary Differential Equation

$$-\beta\frac{dz}{d\xi} + z\frac{dz}{d\xi} + \frac{d^3 z}{d\xi^3} = 0 \tag{3.15}$$

Integrating can be done directly since equation (3.15) is a form of a total derivative. It follows from equation (3.15) that

$$-\beta z + z^2 + \frac{d^2 z}{d\xi^2} = c_1 \tag{3.16}$$

where c_1 is a constant of integration. In order to obtain the first order equation for z multiplication with $\frac{dz}{d\xi}$ is done, so that

$-\beta z\frac{dz}{d\xi} + z^2\frac{dz}{d\xi} + \frac{d^2 z}{d\xi^2}\frac{dz}{d\xi} = c_1\frac{dz}{d\xi}$

$\Rightarrow -\beta z dz + z^3 + \frac{d^2 z}{d\xi^2}dz = c_1 dz$

Integrating both sides (with c_2 as the constant of integration) leads to

$$-\frac{\beta}{2}z^2 + z^3 + \frac{1}{2}(\frac{dz}{d\xi})^2 = c_1 z + c_2 \tag{3.17}$$

Now it is required that when $x \longrightarrow \pm\infty$ we should have $z \longrightarrow 0$, $\frac{dz}{d\xi} \longrightarrow 0$ and $\frac{d^2z}{d\xi^2} \longrightarrow 0$.

From these requirements it follows that $c_1 = c_2 = 0$

Remark (3.1)

More general solutions can be found for other choices of c_1 and c_2. These solutions can be represented in terms of elliptic integrals, for details see Drazin and Johnson [3]

With $c_1 = c_2 = 0$. Equation (3.17) can be written as

$$(\frac{dz}{d\xi})^2 = z^2(\beta - 2z) \tag{3.18}$$

By separation of variables we may write

$$\int_0^z \frac{d\zeta}{\zeta\sqrt{\beta - 2\zeta}} = \int_0^\xi d\eta \tag{3.19}$$

The choice of 0 for lower integration limits does not bring any loss of generality since the starting point can be transformed linearly.

The integration of the left hand side of equation (3.19) can be done by using the transformation

$$s = \frac{1}{2}\beta sech^2 w \tag{3.20}$$

The role of s here is played by the variable ζ and we obtain

$$\beta - 2\zeta = \beta(1 - sech^2 w) = \beta tan^2 w \tag{3.21}$$

since the relation $cosh^2 w - sinh^2 w = 1$ holds.
Furthermore we have

$$\frac{d\zeta}{dw} = -\beta \frac{sinhw}{cosh^3 w} \tag{3.22}$$

The upper limit of the left hand integral in equation (3.19) due to equation (3.20) is transformed to

$$w = sech^{-1}\sqrt{\frac{2z}{\beta}} \tag{3.23}$$

Substituting equations (3.21), (3.22) and (3.23)into equation (3.19) we get

$\zeta = -\frac{2}{\sqrt{\beta}}\int_0^w \frac{1}{sech^2.tanhw}\cdot\frac{sinhw}{cosh^3 w} = -\frac{2}{\sqrt{\beta}}\int_0^w \frac{cosh^2 w.coshw}{sinhw}\cdot\frac{sihw}{cosh^3 w}$

$\Rightarrow \zeta = -\frac{2}{\sqrt{\beta}}\int_0^w dw = -\frac{2}{\sqrt{\beta}}w$

with equation(3.20) the transformation back to ζ is done and we obtain:

$\xi = -\frac{2}{\sqrt{\beta}} sech^{-1}\sqrt{\frac{2z}{\beta}}$

$\Rightarrow z(\xi) = \frac{\beta}{2} sech^2(\frac{\sqrt{\beta}}{2})\xi$

Now we finally use equation (3.14) to get

$$u(x,t) = \frac{\beta}{2} sech^2[\frac{\sqrt{\beta}}{2}(x - \beta t)] \tag{3.24}$$

Remark (3.2):

In order to have a real solution the quantity β must be positive. As it can be seen from equation (3.14) for $\beta > 0$, the solitary wave moves to the right. The second point is that the amplitude is proportional to the speed which is indicated by the value of β. Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves.

To perform superpositions we consider the following:

If, instead of equation (3.20), we select the transformation

$$s = -\frac{1}{2}\beta csch^2 w \tag{3.25}$$

Then in the same way we obtained equation (3.24) we will obtain another solution which is:

$$u(x,t) = \frac{\beta}{2}csch^2[\frac{\sqrt{\beta}}{2}(x-\beta t)] \tag{3.26}$$

The solution of equation (3.26) is an irregular solution to the KdV equation. It has a singularity for vanishing argument of the cosech-function, that is, for the line in the $x-t$ plane, with $x-\beta = 0 \Leftrightarrow t = \frac{1}{\beta}x$

Chapter 4

RESULTS AND CONCLUSION

The original choice was to design and apply the most basic finite difference scheme which could possibly be considered.Using the lowest order approximation to u_x and u_{xxx} respectively, then implement the explicit Euler time-stepping. The original scheme was as follows:

$$\frac{u_j^{n+1}-u_j^n}{h} + u_j^n.(\frac{u_{j+1}^n-u_j^n}{\Delta x}) + (\frac{u_{j+2}^n-2u_{j+1}^n+2u_{j-1}^n-u_{j-2}^n}{(\Delta x^3)}) = 0;$$

When we solve for u_j^{n+1} in order to set up the explicit time step with $r = \Delta t/\Delta x$ and $\Delta t = h$ we get:

$$u_j^{n+1} = u_j^n + \frac{h}{\Delta x}(u_{j+1}^n - u_j^n)u_j^n + \frac{h}{2(\Delta x)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n) \quad (4.1)$$

We solved equation (1.1) with Scheme (4.1) subject to:

$$u(x,0) = 3\alpha^2 sech(\frac{\alpha(x+2)}{2})^2 + 3\beta^2 sech(\frac{\beta x}{2})^2 \quad (4.2)$$

within the domain $[-\pi : \pi]$ and replacing $\Delta x = k$.

We then used a C++ computer program with $sech(x)$ expressed as $\frac{1}{cosh(x)}$ to get:

```
♯include ⟨cmath⟩
♯include ⟨cstdlib⟩
♯include ⟨iostream⟩
♯include ⟨cstdlib⟩
inline double P( double alpha, double beta, double x0, double x1 )
  {double u = cos(alpha(x1 − x0) + 2)/2 ∗ cosh(alpha(x1 − x0) + 2)/2;
  double v = beta ∗ cosh(beta(x1 − x0)/2) ∗ beta ∗ cosh(beta(x1 − x0)/2);
  return ((3 ∗ alpha ∗ alpha ∗ 1/u) + (3 ∗ beta ∗ beta ∗ 1/v));}
inline double KdV( double result, double h, double k)
    double z = h/(2 ∗ k ∗ k);
    return (((1 + h/k ∗ result) ∗ result + (z/k ∗ result − 2 ∗ z ∗ result −
z/k ∗ result))/(1 + h/k ∗ result − z/k));}
main( int argc, char∗ ∗ argu, char∗ ∗ argv, char∗ ∗ argz )
  {double h, result, alpha, beta, k, x0,x1, peg0, peg1;
  std::cout << "Enter the values :" << std::endl;
  std::cout << "parameter alpha :"; std::cin >> alpha;
  std::cout << "parameter beta :"; std::cin >> beta;
  std::cout << "parameter k :"; std::cin >> k;
  std::cout << "step size, h :"; std::cin >> h;
  std::cout << "Enter x0 :"; std::cin >> x0;
  std::cout << "Enter x1 :"; std::cin >> x1;
  for (short n = 0; n < 30; n++)
```

$\{peg0 \;=\; x1;$

$result \;=\; KdV(P(alpha,\; beta,\; x0,\; x1)h,\; k);$

std::cout << $"With\ P"$ << n << $"\;=\;"$ << $result$ <<std::endl;

system($"pause"$);}

The following conditions were taken into account when compiling the results:

1. The best results were obtained when $alpha = 2$, and $beta = 0.015$ so the choice of parameters use was influenced by the given parameters.
2. The stability of the numerical scheme is governed by small step size, h, satisfying Courant-Friedrichs-Lewy (CFL) condition.

This lead to the following tabulated result in table (4.1)

t	u when $k = 0.005$ and $h = 0.0000095$	u when $k = 0.005$ and $h = 0.00001$	u when $k = 0.005$ and $h = 0.000015$
0	-0.000252205	-0.000231554	-0.000102047
0.5	0.10087	0.0924541	0.0396788
1	0.724841	0.749816	-3.2983
1.5	0.114197	0.108879	0.0397296
2	-0.178916	-0.17013	-0.016842
2.5	-0.354364	-0.373543	-1.16842
3	-0.594836	-0.515886	-0.0957764
3.5	-0.432019	-0.75429	0.0677262
4	0.476013	-0.439421	0.216592
4.5	0.0708176	0.247162	0.203257
5	-0.25966	0.103769	-60.0058
5.5	-0.315641	-0.744896	-0.0106603
6	-2.13824	-0.134092	-0.0124892
6.5	-0.076382	0.119589	-4328.41
7	0.0158029	0.306169	5.52929e-009
7.5	0.782475	0.411152	-5.52929e-009
8	0.088365	0.679249	1.20045e+009
8.5	-0.160087	0.28869	0
9	-0.417041	-0.268213	0
9.5	-0.403965	-0.193429	-1.IND
10	-1.73107	0.875247	-1.IND
10.5	-0.73107	0.0581244	-1.IND
11	0.0273974	-0.137828	-1.IND
11.5	0.611981	-0.535434	-1.IND
12	0.123562	-0.265082	-1.IND
12.5	-0.218477	0.288119	-1.IND
13	-0.305327	0.133656	-1.IND
13.5	-'021915	-0.687314	-1.IND
14	-0.121915	-0.135145	-1.IND
14.5	0.05035	0.135145	-1.IND
15	0.450104	0.284531	-1.IND

Table 4.1:Results of the explicit scheme with $h = 0.0000095$, $h = 0.00001$ $h = 0.000015$

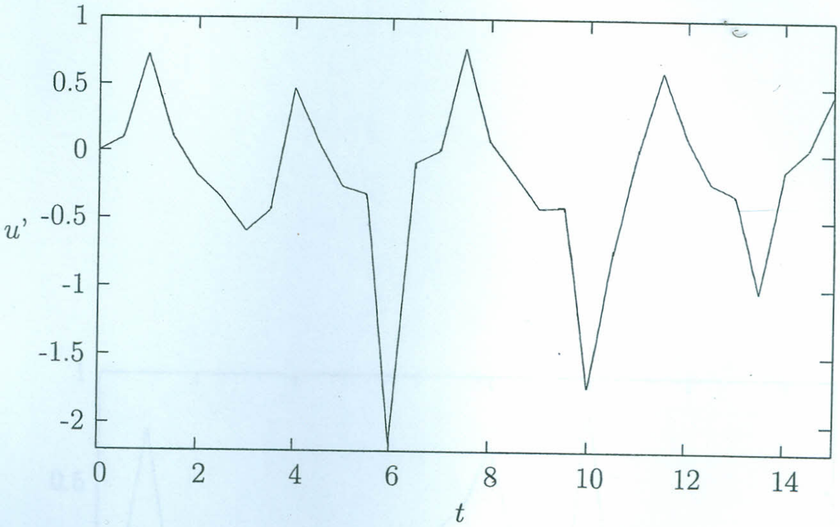

Figure 4.1: Plot for KdV Explicit scheme with $\alpha = 2$,$\beta = 0.015$,$k = 0.005$ and $h = 0.0000095$

The finite difference scheme is therefore conditionally stable and as such, maximum stability will be achieved when $h = 0.00001$. As we move from the maximum stability parameter, the scheme becomes unstable.

4.1 Stability of The Numerical Scheme Employed

From the results of the scheme tabulated above and the corresponding graphical representation:

- The numerical scheme when $h = 0.0000095$, was stable up to the $293rd$ step and could be said to be weakly stable.

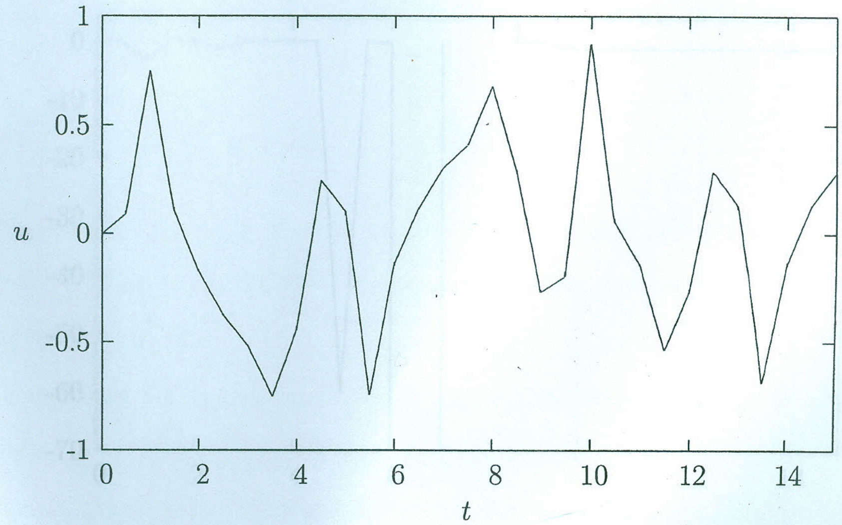

Figure 4.2: Plot for KdV Explicit scheme with $\alpha = 2$,$\beta = 0.015$,$k = 0.005$ and $h = 0.00001$

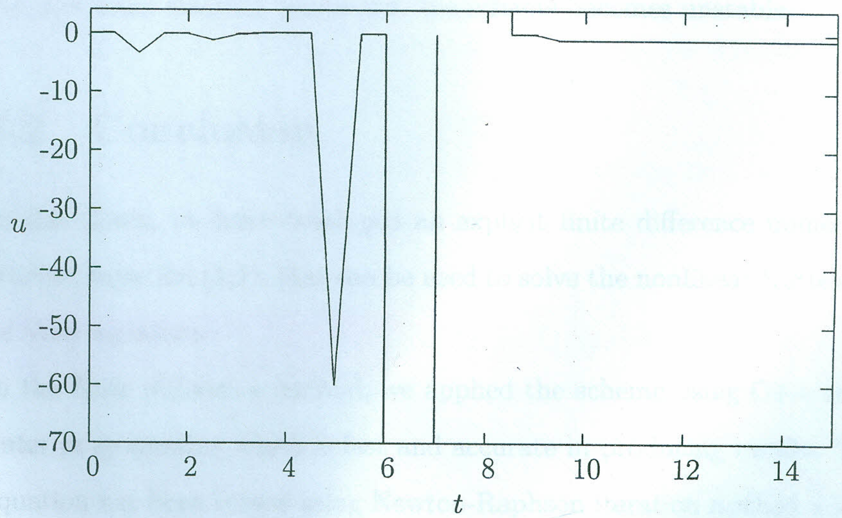

Figure 4.3: Plot for KdV Explicit scheme with $\alpha = 2$,$\beta = 0.015$,$k = 0.005$ and $h = 0.000015$

- The numerical scheme stable when $h = 0.00001$, up to the $846th$ step and could be said to be stable.
- The numerical scheme, when h is increased to $h = 0.000015$ was stable up to the $12th$ step and could be said to be unstable.

The numerical scheme is therefore conditionally stable and as such, maximum stability will be achieved when $h = 0.00001$. As we move from the maximum stability parameter, the scheme becomes unstable.

4.2 Conclusion

In this thesis, we have developed an explicit finite difference numerical scheme, equation (4.1), that can be used to solve the nonlinear Korteweg-de Vries equation.

In the finite difference method, we applied the scheme using C++ computer programming which is fast and accurate in producing results. The equation has been solved using Newton-Raphson iteration method which converges fast to a meaningful solution. The results are tabulated and the graphical plots are given using GNUplot package.

The explicit finite difference scheme developed is stable when $h = 0.00001$. In the three graphical outputs, (figure 4.1, figure 4.2 and figure 4.3), it is clear that figure 4.2 is smooth and uniform as compared with the other graphs and we can say it is therefore an accurate result.

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