ON THE NUMERICAL RANGES AND SPECTRA OF NORMAL OPERATORS

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Abstract

The study of numerical ranges and spectra has been of great interest to many mathematicians in the past decades. In this study, we have continued to look at the numerical ranges and spectra of operators on a Hilbert space. The properties of numerical range, for example, convexity and closedness are well known as proved in the classic Toeplitz - Hausdorff Theorem. In this study, we investigate the relationship between the spectrum and the numerical range of an operator, in particular, when the operator is normal. We have established that for a bounded linear operator on a Hilbert space, the spectrum is contained in the closure of its numerical range. For a normal operator, we have also established that the numerical radius and the spectral radius coincides with the norm of the operator. These results are actually a contribution to the field of numerical ranges and spectra. For us to achieve these, it was paramount that we had a deep understanding of the theory of operators, especially on Hilbert spaces, General Topology, Functional Analysis and Abstract Algebra. This was achieved by reading the available and relevant literature, solving the existing problems and understanding examples in these areas. Further, we also had consultative meetings with the supervisors. In addition, we explored internet Information and further references through the use of research papers in this field. Lastly we could not avoid consultations with other mathematicians who have carried research in this field of study.

Chapter 1

BASIC CONCEPTS

1.1 Introduction

The study of numerical ranges was first carried out and presented originally by Toeplitz in 1918. He proved that the boundary of numerical range for an operator on a Hilbert space is convex [20]. Later, Hausdorff proved that $W(T)$ was simply connected. The work of these two scholars later gave rise to the classic Toeplitz- Hausdorff theorem [16]. The subject aroused a lot of curiosity, and a number of mathematicians have done research in this area over the years. Agure [2] later gave an alternative proof to this theorem (Toeplitz- Hausdorff theorem).

This study is primarily concerned with the numerical range and the spectrum of normal operators on Hilbert space.

The first chapter is composed of basic concepts which we intend to use in subsequent chapters. We also present terminologies and symbols.

In chapter two we discuss properties of the numerical range and examples on how to calculate the numerical range.

In chapter three, we look at the relationship between the spectrum and the closure of its numerical range and further discuss normal operators and the properties of algebraic numerical range. Finally, we give the conclusion and recommendations of our work in chapter four.

First, we need to define certain concepts before we start using them.

Definition 1.1.1. Subspace.

Given a vector space X over a field $\mathbf{K}$, a subset W of X is called a **subspace** if W is a vector space over $\mathbf{K}$ and under the operations already defined on X.

Definition 1.1.2. Algebra.

Let X be a vector space with a field $\mathbf{K}$, an **algebra** is a vector space X together with a bilinear map $X \times X \to X$ defined by $(a,\ b) \to ab \quad \forall,\ a,\ b \in X$ such that $a(bc) = (ab)c \quad \forall,\ a,\ b,\ c \in X$.

Definition 1.1.3. Norm.

Let X be a vector space over $\mathbf{K}$. A function $\|,\| : X \longrightarrow \mathbf{R}$ is called a **norm** if it satisfies the following properties; $\forall,\ a,\ b \in X$ and $\forall,\ \lambda \in \mathbf{K}$

(i) $\|a\| \geq 0$,

(ii) $\|a\| = 0$ iff $a = 0$,

(iii) $\|\lambda a\| = |\lambda| \|a\|$,

(iv) $\|a + b\| \leq \|a\| + \|b\|$.

Definition 1.1.4. Metric space.

Let X be a nonvoid set and $\rho : X \times X \to \mathbb{R}^{+} \bigcup \{0\}$ be a non-negative function satisfying the properties

(i) $\rho(x,\ y) = \rho(y,\ x)$, $\forall\ x,\ y \in X$,

(ii) $\rho(x,\ y) = 0$ if and only if $x = y$,

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z), \forall\, x, y \text{ and } z \in X$.

Then the ordered pair (X, ρ) is called **a metric space.**

Definition 1.1.5. Banach space.

A **Banach space** is a normed space which is a complete metric space.

Definition 1.1.6. Inner product.

Let X be a vector space over $\mathbf{K}$ (the field of real or complex numbers.) A mapping denoted by $\langle .,. \rangle$ defined on $X \times X$ into the underlying field is called an **inner product** of any two elements x and y of X if the following conditions are satisfied:

(i) $\langle x, x \rangle \geq 0, \forall, \ x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(ii) For any x, x' and y of X, $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$,

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ where α belongs to the underlying field,

(iv) $\overline{\langle x, y \rangle} = \langle y, x \rangle$.

Definition 1.1.7. Inner product space.

Let X be a vector space over $\mathbf{K}$ and $\langle .,. \rangle$ be a mapping, $\langle .,. \rangle : X \times X \longrightarrow \mathbf{K}$. Then the pair $(X, \langle .,. \rangle)$ is called an **inner product space** over $\mathbf{K}$.

Definition 1.1.8. Hilbert space.

A **Hilbert space** is a complete inner product space i.e a Banach space whose norm is generated by an inner product.

Definition 1.1.9. Involution.

Let A be an algebra. A mapping from $A \to A$ defined by $x \mapsto x^* \quad \forall, x, x^* \in A$ is called an **involution** on A if it satisfies the following four conditions; $\forall\, x, y \in A$ and λ a scalar,

(i) $(x + y)^* = x^* + y^*$,

(ii) $(\lambda x)^* = \bar{\lambda} x^*$,

(iii) $(xy)^* = y^*x^*$,

(iv) $x^{**} = x$.

Definition 1.1.10. *-algebra.

An algebra A with an involution i.e. $x \mapsto x^*$ is called **a *-algebra**.

Definition 1.1.11. Banach *-algebra.

A Banach *-algebra is a normed algebra A with involution which is complete and has the property that $\|x\| = \|x^*\|$. In this case, we define a normed algebra as follows: i.e. the algebra A is **a normed algebra** if for each element $x \in A$ there is an associated real number $\|x\|$, the norm of x satisfying the axioms of the norm. Thus, $\forall\ x,\ y \in A$,

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$,

(iv) $\|xy\| \leq \|x\|\|y\|$.

Definition 1.1.12. C*-algebra.

A Banach *-algebra A with the property $\|x^*x\| = \|x\|^2$, $\forall\ x \in A$ is called **a C*-algebra**.

Definition 1.1.13. Basis.

A **basis** S for a vector space X is a nonempty set of linearly independent vectors that span X.

Definition 1.1.14. Orthonormal basis.

Let $(X, \langle .,. \rangle)$ be an inner product space. Then, $\forall,\ x,\ y\ \in X$, x and y are said to be orthonormal if $\langle x,\ y \rangle = 0$ and $\|x\| = \|y\| = 1$. An orthonormal set of all vectors of the form x and y which form a basis is called an **orthonormal basis**.

Definition 1.1.15. Operator.

An **operator** is a mapping of a vector space X onto itself or to another vector space.

Definition 1.1.16. Linear Operator.

Let X and Y be vector spaces. Then a function $T : X \to Y$ is called **a linear operator** if and only if, $\forall\ x_1, x_2 \in X$ and $\forall\ \lambda,\ \mu \in \mathbf{K}$, $T(\lambda x_1 + \mu x_2) = \lambda T(x_1) + \mu T(x_2)$.

Definition 1.1.17. Bounded linear Operator.

Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is called **a bounded linear operator** if and only if there exists a constant $M > 0$ such that, $\|Tx\| \leq M\|x\|,\ \forall\ x \in X$.

Definition 1.1.18. Adjoint of T.

If $T \in B(H, K)$, where H, K are Hilbert spaces, then the unique linear operator $T^* \in B(K, H)$ satisfying $\langle Tx,\ y\rangle = \langle x,\ T^*y\rangle$, $\forall$, $x \in H$ and $y \in K$ is called the **Adjoint** of T.

Definition 1.1.19. Self - adjoint operator. A bounded operator $T \in B(H)$ is said to be self- adjoint if $T = T^*$. Thus T is Hermitian and $D(T) = H$ if and only if T is self - adjoint.

Definition 1.1.20. Normal operator.

A bounded linear operator T on a Hilbert space H is said to be a **normal operator** if it commutes with its adjoint, that is $TT^* = T^*T$.

Definition 1.1.21. Unitary operator.

A **unitary operator** is a bounded linear operator U on a Hilbert space satisfying: $U^*U = UU^* = I$, where I is the identity operator.

This property implies the following:

(i) U preserves inner product on the Hilbert space, so that for all vectors x and y in the Hilbert space H, $\langle Ux,\ Uy\rangle = \langle x,\ y\rangle$.

Proof.

$$\begin{aligned}\langle Ux, Uy\rangle &= \langle x, U^*Uy\rangle \\ &= \langle x, Iy\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

□

(ii) U is a surjective isometry (distance preserving map) i.e

$$\|U(x-y)\| = \|x-y\|.$$

Proof.

$$\begin{aligned}\|U(x-y)\|^2 &= \langle U(x-y), U(x-y)\rangle \\ &= \langle (x-y), U^*U(x-y)\rangle \\ &= \langle (x-y), I(x-y)\rangle \\ &= \langle (x-y), (x-y)\rangle \\ &= \|(x-y)\|^2 \\ \Rightarrow \|U(x-y)\| &= \|(x-y)\|.\end{aligned}$$

□

Definition 1.1.22. Compact operator.

If H is a Hilbert space, then an operator $T \in B(H)$ is a finite **rank**

operator if the dimension of the range of T is finite and a **compact operator** if for every bounded sequence $(x_n) \in H$, the sequence (Tx_n) contains a convergent subsequence.

Definition 1.1.23. Functional.

A **functional** is a mapping of a vector space into a field of scalars **K** ($\mathbb{R}$ or $\mathbb{C}$).

Definition 1.1.24. Linear functional.

$f : X \to \mathbb{C}$ is **a linear functional** on X if f is a linear operator, that is, a linear functional is a complex-valued linear operator.

Definition 1.1.25. Bounded linear functional.

A linear functional f is called **a bounded linear functional** if and only if there exists a constant $N > 0$ such that, $|f(x)| \leq N\|x\|, \ \forall x \in X$.

Definition 1.1.26. Positive linear functional.

A positive linear functional is a linear functional on a Banach algebra A with an involution that satisfies the condition

$$f(xx^*) \geq 0, \ \forall, \ x \in A.$$

Definition 1.1.27. State.

Let A be an algebra with involution. Then the linear functional f is called **a state** on A if f is positive and $\|f\| = f(e) = 1$, where e is an identity element in A .

Definition 1.1.28. Eigenvalue.

Let H be a Hilbert space and $T : H \longrightarrow H$ a linear operator. For any $T \in B(H)$ a number $\lambda \in \mathbb{C}$ is called the **eigenvalue of** T if there is a non-zero $x \in H$ such that $Tx = \lambda x$, the vector x is then called **an eigenvector for** T corresponding to the eigenvalue λ.

Definition 1.1.29. Convex set.

Let X be a linear space. A subset M of the linear space X is **convex** if $\forall$, x, $y \in M$, and for any positive real number t satisfying $0 < t < 1$, we have $tx + (1-t)y \in M$.

Definition 1.1.30. Convex hull.

If M is a subset of a linear space X, then **a convex hull** of M, represented by $conv(M)$ is the smallest convex subset of X containing M, that is the intersection of all the convex subsets of X that contain M.

Definition 1.1.31. Numerical range of T.

Let H be a Hilbert space and $T : H \to H$ be a linear operator. For any $T \in B(H)$, **the numerical range** is the set defined as

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

Note: The numerical range $W(T)$ has the following **properties**:

(i) $W(T)$ is non-empty.

(ii) $W(T)$ is unitarily invariant.

That is, $W(U^*TU) = W(T)$, U is unitary operator on H.

(iii) $W(T)$ lies in the closed disc of radius $\|T\|$ centered at the origin.

(iv) $W(T)$ contains all the eigenvalues of T that is, $\lambda \in W(T)$.

(v) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$.

(vi) $W(I) = \{1\}$, I is the identity of $B(H)$.

(vii) If α, β are complex numbers, and T a bounded linear operator on H, then $W(\alpha T + \beta I) = \alpha W(T) + \beta$.

(viii) If H is finite dimensional then $W(T)$ is compact.

(ix) $W(T)$ is a convex set (the Toeplitz-Hausdorff Theorem).

Definition 1.1.32. Spectrum of T.

For any $T \in B(H)$,

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \quad \text{is not invertible in} \quad B(H)\}$$

is called the **spectrum** of T.

Definition 1.1.33. Spectral radius.

Let H be a Hilbert space and $T : H \to H$ be a linear operator. The number

$$\gamma(T) = \sup\{| \lambda | : \lambda \in \sigma(T)\}$$

is called the **spectral radius** of T.

Definition 1.1.34. Numerical radius.

Let H be a Hilbert space and $T : H \to H$ be a linear operator. The number

$$w(T) = \sup\{| \lambda | : \lambda \in W(T)\}$$

is called the **numerical radius** of T.

1.2 Literature review

For a normal operator T on a Hilbert space H, the numerical range $W(T)$ has a definition which was originally introduced for finite dimensional spaces by Toeplitz [20] in 1918. He proved that, the boundary of numerical range $\partial W(T)$ for an operator on a Hilbert space is convex [20]. Later, Hausdorff proved that the set $W(T)$ is simply connected. The work of these two scholars later gave rise to the classic Toeplitz- Hausdorff theorem [16]. The subject aroused a lot of curiosity, and a number of mathematicians have done research in this area over the years.

Agure [1] introduced a strong Toeplitz - Hausdorff property for the operator $T \in B(H)$ and established the necessary and sufficient condition for the set $W(T)$ to be convex. In [2] he went on to give an alternative proof to the classical Toeplitz - Hausdorff theorem . Stampfli [19] later introduced the sets $W_0(T)$ and $W_\delta(T)$, the maximum numerical range and the δ-numerical range respectively , given by

$$W_0(T) = \{\lambda : \langle Tx_n,\ x_n\rangle \to \lambda, \quad \|x_n\| = 1, \quad \|Tx_n\| \to \|T\|\}.$$

and

$$W_\delta(T) = closure\{\langle Tx,\ x\rangle : \ x \in H, \ \|x\| = 1, \quad \|Tx\| \geq \delta\}.$$

When H is finite dimensional, $W_0(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$).

In [19] he proved the convexity for $W_0(T)$. In [2], Agure showed that

$W_\delta(T)$ for any $T \in B(H)$ is convex.

For an algebra A and $T \in A$, we can define the algebraic numerical range $V(T)$ for an operator T as $V(T) = \{f(T) : f \in E(A)\}$ where $E(A)$ is the set of states on A.

Agure in [1] introduced the algebra δ-numerical range which he defined as $V_\delta(T) = \{f(T) : f(I) = \|f\| = 1, \ f(T^*T) \geq \delta^2\}$ and showed that $W_\delta(T) = V_\delta(T)$ for all $T \in B(H)$.

Therefore, the purpose of our study was to further investigate the set $W(T)$ for a normal operator T and find out if there is a relationship between numerical range and the spectrum $\sigma(T)$.

1.3 Statement of the problem

Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. For any $T \in B(H)$, the sets $W(T)$ and $\sigma(T)$ denote the numerical range and the spectrum of T respectively. In this study, we investigate the relationship between the spectrum $\sigma(T)$ and the numerical range $W(T)$, specifically when T is normal. We further investigate certain properties of normal operators and the algebra numerical range.

1.4 Objective of the study

The main purpose of this study is to investigate the relationship between numerical range and the spectrum of T, in particular when T is normal.

1.5 Research methodology.

In order to make a significant progress in this work, it was essential to have a deep understanding of the theory of operators, especially on Hilbert Spaces, and Functional Analysis. This was achieved by reading the available and relevant literature, solving the existing problems and understanding examples in these areas.

There was also need to have consultative meetings with the supervisors. Information from the internet became useful. Consultation with other mathematicians who have done research in this field was of great help.

Chapter 2

NUMERICAL RANGES

2.1 Introduction

In this chapter, we shall be interested in bounded linear operators on a complex Hilbert space H. Here, we see that, the numerical range $W(T)$ of any operator $T \in B(H)$ such that $T : H \longrightarrow H$ is the subset of the complex numbers $\mathbb{C}$ given by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

This is often called the field of values.

We shall now look at some properties of this set and give their proofs and further consider some examples.

2.2 Properties of numerical range

The set $W(T)$ has several interesting properties for $T \in B(H)$.

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ *for* $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$.

Proof.

$$\begin{aligned}
W(\alpha I + \beta T) &= \{\langle(\alpha I + \beta T)x, x\rangle : x \in H, \|x\| = 1\} \\
&= \{\langle \alpha I x, x\rangle + \langle \beta T x, x\rangle : x \in H, \|x\| = 1\} \\
&= \{\alpha\langle I x, x\rangle + \beta\langle T x, x\rangle : x \in H, \|x\| = 1\} \\
&= \{\alpha\langle x, x\rangle + \beta\langle T x, x\rangle : x \in H, \|x\| = 1\} \\
&= \{\alpha\|x\|^2 + \beta\langle T x, x\rangle : x \in H, \|x\| = 1\} \\
&= \alpha + \beta\{\langle T x, x\rangle : x \in H, \|x\| = 1\} \\
&= \alpha + \beta W(T).
\end{aligned}$$

□

(ii) $W(T^*) = \{\overline{\lambda} :\ \lambda \in W(T)\}$.

Proof.

$$\begin{aligned} W(T^*) &= \{\langle T^*x, x\rangle : x \in H, \|x\| = 1\} \\ &= \{\langle x, Tx\rangle : x \in H, \|x\| = 1\} \\ &= \left\{\overline{\langle Tx, x\rangle} : x \in H, \|x\| = 1\right\} \\ &= \{\overline{\lambda} : \lambda \in W(T)\}. \end{aligned}$$

□

(iii) $W(U^*TU) = W(T)$, for any unitary U.

Proof.

$$\begin{aligned} W(U^*TU) &= \{\langle U^*TUx, x\rangle : x \in H, \|x\| = 1\} \\ &= \{\langle TUx, U^{**}x\rangle : x \in H, \|x\| = 1\} \\ &= \{\langle TUx, Ux\rangle : x \in H, \|x\| = 1\} \\ &= \{\langle Ty, y\rangle : y \in H, \|y\| = \|Ux\| = \|x\| = 1\} \quad (Ux = y) \\ &= W(T). \end{aligned}$$

□

(iv) $W(T)$ lies in a closed disc of radius $\|T\|$ centered at origin.

Proof. Let $\lambda \in W(T)$ then, $\exists\, x \in H$ with $\|x\| = 1$ such that

$$\begin{aligned} |\lambda| &= |\langle Tx, x\rangle| \\ &\leq \|Tx\|\|x\| \\ &\leq \|T\|\|x\|^2 \\ &= \|T\|. \end{aligned}$$

Thus $W(T) \subseteq \overline{N}(\overline{0}, \|T\|)$ which is a closed disc centered at the origin with radius $\|T\|$. This completes the proof. □

(v) $W(T)$ contains all eigenvalues of T.

Proof. Let $Tx = \lambda x$ with $\|x\| = 1$ then for all x,

$$\begin{aligned} \langle Tx, x\rangle &= \langle \lambda x, x\rangle \\ &= \lambda\langle x, x\rangle \\ &= \lambda\|x\|^2 \\ &= \lambda. \end{aligned}$$

$\Rightarrow \lambda \in W(T)$. □

(vi) $W(I) = \{1\}$.

Proof.

$$\begin{aligned} W(I) &= \{\langle Ix, x\rangle \ : x \in H, \ \|x\| = 1\} \\ &= \{\langle x, x\rangle \ : x \in H, \ \|x\| = 1\} \\ &= \{\|x\|^2 \ : x \in H, \ \|x\| = 1\} \\ &= \{1\}. \end{aligned}$$

□

(vii) $W(T)$ is convex.

This property of numerical range forms the backbone of our study. The convexity of $W(T)$ has been proved in more than one way by a number of scholars for example, Agure [2] and Toeplitz [20] among others. In this study, we shall provide an alternative proof to this property which is much simpler and more direct.

But we shall first prove the following two basic Lemmas which clearly presents the structure of the numerical range for a 2-dimensional Hilbert space, and at the same time shall be used in our proof. The first Lemma is the following;

Lemma 2.2.1. *Let T be a linear operator on a 2-dimensional Hilbert space ℓ_2. If the matrix of T which is a 2 X 2 matrix has distinct eigenvalues λ_1 and λ_2 and the corresponding eigenvectors x_1 and x_2, so normalized such that $\|x\| = \|y\| = 1$, then $W(T)$ is a closed elliptic disc with foci at λ_1 and λ_2.*

If $\gamma = |\langle x_1, x_2\rangle|$ and $\delta = \sqrt{1-\gamma^2}$ then the minor axis is $\gamma|\lambda_1 - \lambda_2|/\delta$ and the major axis is $|\lambda_1 - \lambda_2|/\delta$.

If T has only one eigenvalue λ, then $W(T)$ is the circular disc with *center at λ, and radius $\frac{1}{2}\|T - \lambda I\|$.*

Proof. Since ℓ_2 has unit disc $\{x : \|x\| = 1\}$ as a compact set and the function $x \longmapsto \langle Tx, x\rangle$ is continuous, it follows that $W(T)$ is a compact set.

Suppose T has only one eigenvalue λ.

In this case $T_1 = T - \lambda I$ has the property that $\sigma(T_1) = \{0\}$, and also $T_1^2 = 0$ for the characteristic polynomial of the matrix T is $p(t) = \alpha(t - \lambda)^2$, for non-zero $\alpha \in \mathbb{C}$. Hence $\alpha(T - \lambda I)^2 = 0$, i.e $T_1^2 = 0$. If $T_1 = 0$, we have $W(T_1) = \{0\}$, and thus $W(T) = \{\lambda\}$.

This clearly is a circle with center λ and radius 0. If $T_1 \neq 0$, then there exists an orthonormal basis $\{e_1, e_2\}$ of ℓ_2 such that $T_1 e_1 = a e_2$, $T_1 e_2 = \bar{0}$ and $\|T_1\| = |a|$.

This implies that $W(T_1)$ is a closed circular disc with centre λ and radius $= \frac{|a|}{2} = \frac{\|T_1\|}{2} = \frac{\|T-\lambda I\|}{2}$.

Now if T has distinct eigenvalues λ_1 and λ_2, the operator

$$T_1 = \frac{1}{\lambda_2 - \lambda_1}(T - \lambda_1 I)$$

has eigenvalues 0 and 1.

Let $\{e_1, e_2\}$ be an orthonormal basis for ℓ_2 such that and we choose this such that

$$T_1 u = u, \quad \|u\| = 1$$

where $u = (\cos\varphi)e_1 + (\sin\varphi)e_2$ and φ is the angle between u and

e_1, that is, $\cos\varphi = |\langle e_1, u\rangle|$, $0 \le \varphi \le \frac{\pi}{2}$. Now since $T_1 u = u$, we have

$$\begin{aligned} T_1(\cos\varphi e_1 + \sin\varphi e_2) &= \cos\varphi e_1 + \sin\varphi e_2 \\ &= \sin\varphi e_2 \\ \sin\varphi T_1 e_2 &= \cos\varphi e_1 + \sin\varphi e_2 \\ T_1 e_2 &= \cot\varphi e_1 + e_2 \end{aligned}$$

Now take any $x = ae_1 + be_2$, $\|x\| = 1$ with $|a|^2 + |b|^2 = 1$.

Then

$$\langle T_1 x, x\rangle = \overline{a}b + |b|^2 = |b|^2 + |a||b|e^{iw}\cot\varphi.$$

If w varies with $|a|$, $|b|$ fixed and $|a|^2 + |b|^2 = 1$, then the scalars $\langle T_1 x, x\rangle$ trace a circle with center at $(t, 0)$ with radius $[t(1-t)]^{\frac{1}{2}}\cot\varphi$ where $t = |b|^2$ and $W(T_1)$ is the union of all the circles.

$$(x-t)^2 + y^2 = (t - t^2)\cot^2\varphi.$$

The envelope of this family of circles is obtained by the equation

$$(2x + \cot^2\varphi)^2 - 4(\csc^2\varphi)(x^2 + y^2) = 0$$

which can be simplified to

$$\frac{(x-\frac{1}{2})^2}{(\frac{1}{2}\csc\varphi)^2} + \frac{y^2}{(\frac{\cot\varphi}{2})^2} = 1$$

This is an ellipse with foci at $(0, 0)$ and $(1, 0)$ and with eccentricity $\sin\varphi$. The center of this ellipse is the point $(\frac{1}{2}, 0)$ and its major and minor axes of lengths $\csc\varphi$ and $\cot\varphi$ respectively. □

The next Lemma is famously known as the ellipse Lemma which demonstrates when foci of an ellipse coincides with the eigenvalues.

Lemma 2.2.2. *(**Ellipse lemma**) Let T be an operator on a two-dimensional Hilbert space. Then $W(T)$ is an ellipse whose foci are the eigenvalues of T.*

Proof. We can choose T such that

$$T = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$$

with λ_1 and λ_2 as the eigenvalues of T.

Now if $\lambda_1 = \lambda_2 = \lambda$, we have

$$T - \lambda I = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

Let $x = (x_1, x_2)$ then,

$$(T - \lambda I)x = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} \|T - \lambda I\| &= \sup\{\|a(x_2, 0)\| : |x_1|^2 + |x_2|^2 = 1\} \\ &= |a|. \end{aligned}$$

Hence the radius is $\frac{1}{2}|a|$. Therefore the numerical range

$$W(T) = \left\{ z : |z| \leq \frac{|a|}{2} \right\}.$$

It thus follows that $W(T)$ is a circle with center at λ and radius $\frac{|a|}{2}$.
Now if $\lambda_1 \neq \lambda_2$ and $a = 0$ we have

$$T = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

If $x = (x_1, x_2)$, then

$$Tx = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{bmatrix}.$$

Therefore taking the inner product $\langle Tx, x \rangle$ we get

$$\langle Tx, x \rangle = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \overline{x_1} + \lambda_2 x_2 \overline{x_2} \end{bmatrix} = \begin{bmatrix} \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 \end{bmatrix}.$$

So

$$\langle Tx, x \rangle = \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2.$$

Now letting $t = |x_1|^2$, we therefore write the above equation as follows $\langle Tx, x \rangle = t\lambda_1 + (1-t)\lambda_2$ since $|x_1|^2 + |x_2|^2 = 1$
So $W(T)$ is the set of convex combinations of λ_1 and λ_2 and is the segment joining them.
If $\lambda_1 \neq \lambda_2$ and $a \neq 0$ we choose λ such that it lies between λ_1 and

λ_2. We therefore have

$$T - \frac{\lambda_1 + \lambda_2}{2} I = \begin{bmatrix} \frac{\lambda_1 - \lambda_2}{2} & a \\ 0 & \frac{\lambda_2 - \lambda_1}{2} \end{bmatrix}$$

In this case, we let $z = re^{-i\Theta}$, $\frac{\lambda_1 - \lambda_2}{2} = re^{-i\Theta}$ and $\frac{\lambda_2 - \lambda_1}{2} = -re^{-i\Theta}$.

So

$$e^{-i\Theta}\left[T - \frac{\lambda_1 + \lambda_2}{2}\right] = \begin{bmatrix} r & ae^{-i\Theta} \\ 0 & -r \end{bmatrix} = T'$$

Here we see that $W(T')$ is an ellipse with center at $(0,0)$ and the minor axis $|a|$, and foci at $(r,0)$ and $(-r,0)$.

Thus, the $W(T)$ is an ellipse with foci at λ_1, λ_2 and the major axis has an inclination of Θ with the real axis. □

We refer the reader to [16] for details on the above two Lemmas. We now proceed to prove the property (vii) above.

Proof. Let a and b be distinct points in $W(T)$ then there exists $x, y \in H$ such that

$$a = \langle Tx, x\rangle, \; b = \langle Ty, y\rangle, \;\; \|x\| = \|y\| = 1.$$

Now let M be the subspace $[\{x, y\}]$ spanned by x and y. Hence M is a closed linear subspace of H of dimension 2 over $\mathbb{C}$.

Assume to the contrary that $\{x, y\}$ is linearly dependent over $\mathbb{C}$, so that $x = \alpha y$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. We then have $\langle Tx, x\rangle = \langle T\alpha y, \alpha y\rangle$

$$\langle Tx, x\rangle = \langle T\alpha y, \alpha y\rangle = |\alpha|^2\langle Ty, y\rangle = \langle Ty, y\rangle.$$

Thus $a = b$ which is a contradiction. Hence $\{x, y\}$ must be linearly independent over $\mathbb{C}$.

Let E be the orthogonal projector on H onto M. Take $z \in M$ with $\|z\| = 1$ we have $Ez = z$ thus $TEz = Tz$

Now Tz need not be in M. However, $ETz \in M$. Consequently $ETEz = ETz$

Thus

$$\langle ETEz, z\rangle = \langle ETz, z\rangle = \langle Tz, Ez\rangle = \langle Tz, z\rangle.$$

Now $\langle Tz, z\rangle \in W(T)$ and we thus obtain $W(ETE) \in W(T)$.

Thus from Lemma 2.2.1 and 2.2.2, since W(ETE) is an ellipse (or circular) disc it follows that $W(T)$ is convex. □

2.3 Examples

The following examples, give elaborate illustrations on how to calculate the field of values that we refer to as numerical range of any given operator T on a finite dimensional Hilbert space H. We note that examples 2.3.1 and 2.3.3 can also be found in [16]. Recall that the numerical range $W(T)$ of an operator T is the subset of the complex numbers $\mathbb{C}$.

Example 2.3.1. In $\mathbb{C}^2$ let T be the operator defined by the matrix

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Take $x \in \mathbb{C}^2$, $x = (f, g)$, $\|x\|^2 = |f|^2 + |g|^2 = 1$ with $\|x\| = 1$.

$$Tx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

and

$$\langle Tx, x \rangle = \begin{bmatrix} g & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = g\overline{f}.$$

Taking absolute values on both sides we have

$$|\langle Tx, x \rangle| = |f||g| = \frac{1}{2}(|f|^2 + |g|^2) = \frac{1}{2}.$$

So $W(T) \subset \left\{z : |z| \leq \frac{1}{2}\right\}$, a circle of radius $\frac{1}{2}$ centered at $(0,0)$. Alternatively, given the operator T defined by the matrix

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we then have the characteristic polynomial given by

$$T - \lambda I = \begin{bmatrix} 0 - \lambda & 1 \\ 0 & 0 - \lambda \end{bmatrix}$$

and hence finding the characteristic equation we see that $\lambda^2 = 0$. Therefore, $\lambda = 0$ is the eigenvalue. Since for the norm we have $\frac{1}{2}\|T\|$ and therefore normalizing the vector x we see that $\|(\frac{x}{\|x\|})\| = 1$.

Now we have $T(f, g) = (g, 0)$. That is

$$Tx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

This implies that $\|T(f,g)\| = \|(g,0)\| = \|g\|$.

From the definition of an operator norm,

$$\begin{aligned} \|T\| &= \sup\{\|T(f,g)\| : \|(f,g)\| = 1\} \\ &= \sup\{\|T(f,g)\| : \sqrt{f^2+g^2} = 1\} \\ &= \sup\{\|g\| : f^2+g^2 = 1\} \\ &= 1. \end{aligned}$$

Therefore, $\frac{1}{2}\|T\| = \frac{1}{2}(1) = \frac{1}{2}$.

Therefore, $W(T)$ is a circle of radius $\frac{1}{2}$ centered at zero.

Example 2.3.2. Let T be the unilateral shift on ℓ_2 of square summa sequences. For any $x \in \ell_2$, $x = (x_1, x_2, x_3, ...)$, with $\|x\| = 1$ and

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty,$$

the unilateral right shift operator $T : \ell_2 \to \ell_2$ is given by

$$Tx = (0, x_1, x_2, x_3, ...) .$$

Now

$$\langle Tx, x\rangle = \left\langle \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ . \\ . \\ . \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix} \right\rangle$$
$$= 0(\overline{x_1}) + x_1\overline{x_2} + x_2\overline{x_3} + ...$$
$$= x_1\overline{x_2} + x_2\overline{x_3} + ...$$

Now, $(|x_1| - |x_2|)^2 \geq 0$ which by the arithmetic - geometric mean inequality implies that $|x_1|^2 + |x_2|^2 \geq 2|x_1||x_2|$.
Similarly, $|x_2|^2 + |x_3|^2 \geq 2|x_2||x_3|$.
Also $|x_3|^2 + |x_4|^2 \geq 2|x_3||x_4|$, and so on. Therefore adding all the terms on the left and similarly on the right of the above equations, we obtain $|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + ... \geq 2|x_1||x_2| + 2|x_2||x_3| + ...$
We thus have

$$\begin{aligned} |\langle Tx, x\rangle| &\leq |x_1\overline{x_2}| + |x_2\overline{x_3}| + ... \\ &= |x_1||\overline{x_2}| + |x_2||\overline{x_3}| + ... \\ &= |x_1||x_2| + |x_2||x_3| + ... \\ &= \frac{1}{2}(2|x_1||x_2| + 2|x_2||x_3| + ...). \end{aligned}$$

Now since $\|x\| = |x_1|^2 + |x_2|^2 + ... = 1$, we have

$$
\begin{aligned}
|\langle Tx, x\rangle| &= \frac{1}{2}[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + ...] \\
&= \frac{1}{2}[(|x_1|^2 + |x_2|^2 + |x_3|^2 + ...) + (|x_2|^2 + |x_3|^2 + ...)] \\
&= \frac{1}{2}[1 + (|x_2|^2 + |x_3|^2 + ...)] \\
&= \frac{1}{2}[1 + (1 - |x_1|^2)] \\
&= \frac{1}{2}[2 - |x_1|^2]
\end{aligned}
$$

If $|x_1| \neq 0$ we see that $|\langle Tx, x\rangle| < 1$. For if $|x_1| = 0$ and x contains a finite number of nonzero entries, we have $|\langle Tx, x\rangle| < 1$ if we consider a minimum natural number n such that $x_n \neq 0$.

Therefore, $W(T)$ is an open disc of radius < 1.

Example 2.3.3. Let the transformation $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be represented by

$$T = \begin{bmatrix} r & b \\ 0 & -r \end{bmatrix}, \; r \in \mathbb{R}, \; b \in \mathbb{C},$$

so that

$$T - \lambda I = T_\lambda = \begin{bmatrix} r - \lambda & b \\ 0 & -r - \lambda \end{bmatrix}$$

and $-(r - \lambda)(r + \lambda) = 0$

$\Rightarrow r^2 - \lambda^2 = 0$

$\Rightarrow r^2 = \lambda^2$.

Therefore $r = \pm\lambda$.

When $r = \lambda$ and given that $(T - \lambda I)x = \overline{0}$, we have

$$(T-\lambda I)x = \begin{bmatrix} 0 & b \\ 0 & -2r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} bx_2 \\ -2rx_2 \end{bmatrix} = x_2 \begin{bmatrix} b \\ -2r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore this implies $x_2 = 0$ and the eigenvectors are of the form $(x_1, 0)$ and eigenvalues are $(1, 0)$.

When $\lambda = -r$, we have

$$T_\lambda x' = \begin{bmatrix} 2r & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 2rx'_1 + bx'_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $x'_1 = \frac{-bx'_2}{2r}$, so the eigenvectors are of the form (x'_1, x'_2).

Therefore $(\frac{-bx'_2}{2r}, x'_2) = x'_2(\frac{-b}{2r}, 1)$. Now let $x'_2 = 1$, the eigenvector is $(\frac{-b}{2r}, 1)$ and the eigenvalues $\frac{1}{\sqrt{4r^2+b^2}}(-b, 2r)$.

2.4 Further results on numerical range

The first result in this section is the following,

Theorem 2.4.1. *$T \in B(H)$ is self-adjoint if and only if $W(T)$ is real.*

Proof. If T is self-adjoint, we have for all $x \in H$,

$$\begin{aligned} \langle Tx, x \rangle &= \langle x, Tx \rangle \\ &= \overline{\langle Tx, x \rangle} \end{aligned}$$

and hence $W(T)$ is real.

Conversely, if $\langle Tx, x \rangle$ is real for all $x \in H$, we have $\langle Tx, x \rangle - \langle x, Tx \rangle = 0$, and so $\langle (T - T^*)x, x \rangle = 0$.

Thus the operator $T - T^*$ has only $\{0\}$ in its numerical range. So this must be a null operator. Therefore, $T - T^* = 0$ and $T = T^*$. □

The next result which is the last in this chapter can also be found in [16] but the proof presented is quite simple and more direct.

Theorem 2.4.2. *Let T be self-adjoint and $W(T)$ is equal to the real interval $[m, M]$. Then $\|T\| = \sup\{|m|, |M|\}$.*

Proof. T is self-adjoint and we can define m and M respectively as

$$m = \inf\{\langle Tx, x\rangle : \|x\| = 1\},$$

and

$$M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}.$$

Therefore when we take the norm of T, we get

$$\|T\| = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$$

which is the result and this gives $\|T\| = \sup\{|m|, |M|\}$. □

Chapter 3

SPECTRA

3.1 Introduction

In this chapter, we discuss the spectrum for a bounded linear operator $T \in B(H)$, denoted by $\sigma(T)$ and give exhaustively its properties. We further explore properties of normal operators and show their relationship with the spectrum. We then establish the relationship between the spectrum and the closure of numerical range. Finally, we extend our study to include some basic properties of the algebra numerical range.
For the definition of the spectrum, see definition 1.1.32.

The spectrum can be separated into three disjoint component sets, namely,

(i) The **point spectrum** which consists of the eigenvalues of T and is defined by

$$P\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not } 1-1 \}.$$

Alternatively, if $\lambda I - T$ could be one-to-one but still not be bounded below, such λ is called **approximate point spectrum** $\sigma_{app}(T)$.

(ii)The **residual spectrum** which is a set defined by

$$R\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is } 1-1 \text{ but, } \mathbf{R}_{(\lambda I - T)} \text{ is not dense } \}.$$

(iii) The **continuous spectrum** $\Gamma\sigma(T)$ which is a set given by

$$\begin{aligned}\Gamma\sigma(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \,\text{is}\, 1-1,\, \mathbf{R}_{(\lambda I - T)} \,\text{is dense,} \\ &\quad (\lambda I - T)^{-1} \,\text{is not continuous on}\, \mathbf{R}_(\lambda I - T)\end{aligned}$$

So $\sigma(T) = P\sigma(T) \cup R\sigma(T) \cup \Gamma\sigma(T)$.

3.2 Properties of the spectrum.

We shall now give the properties of the spectrum in the following remark.

Remark 3.2.1. If $T \in B(H)$, it is known that

(i) $\sigma(T)$ is nonvoid.

(ii) $\sigma(T)$ is closed in $(\mathbb{C}, d)$. (Where $(\mathbb{C}, d)$ is metric space with metric d).

(iii) $\sigma(T) \subseteq \overline{\mathbb{N}}(0, \|T\|)$. (Where $\overline{\mathbb{N}}(0, \|T\|)$ is closed neighbourhood of 0 with radius $\|T\|$).

(iv) The spectral radius, $\gamma(T) = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}}$, $\forall n \in \mathbb{N}$.

Details on remark 3.2.1 can be found in any Functional Analysis book but for this study, we refer to [14].

The next two Propositions characterizes the non-emptiness of the spectrum and the boundedness of the spectral radius, and for details we refer to [3].

Proposition 3.2.2. *Let H be a real Hilbert space and $T \in B(H)$ be self-adjoint. Then $\sigma(T) \neq \emptyset$.*

Proof. For a self-adjoint T, $\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H \text{ and } \|x\| = 1\}$. Then there is a sequence of unit vectors (x_n) of elements of H such that $\|x_n\| = 1, \ \forall \ n \in \mathbb{N}$, and $\langle Tx_n, x_n\rangle \longrightarrow \|T\|$ or $\langle Tx_n, x_n\rangle \longrightarrow -\|T\|$. In the first case, it follows that

$$\begin{aligned} \|(\|T\|I - T)x_n\|^2 &= \|T\|^2\|x_n\|^2 - 2\|T\|\langle Tx_n, x_n\rangle + \|Tx_n\|^2 \\ &\leq \|T\|^2 - 2\|T\|\langle Tx_n, x_n\rangle + \|T\|^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Similarly, in the second case, $\|(\|T\|I + T)x_n\|^2 \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, $\|T\| \in \sigma(T)$ in the first case and $-\|T\| \in \sigma(T)$ in the second case. Thus, $\sigma(T) \neq \emptyset$. □

Proposition 3.2.3. *For any operator $T \in B(H)$, $\gamma(T) \leq \|T\|$.*

Proof. By Remark 3.2.1(iv), we have

$$\begin{aligned} \gamma(T) &= \inf\{\|T^n\|^{\frac{1}{n}} : n \in \mathbb{N}\} \\ &= \lim_{n \longrightarrow \infty} \{\|T^n\|^{\frac{1}{n}}\} \\ &\leq \|T\|. \end{aligned}$$

Therefore $\gamma(T) \leq \|T\|$

Thus $\sigma(T) \subseteq \overline{\mathbb{N}}(0, \|T\|)$. □

We now proceed to give certain results on the spectra and the numerical range.

Theorem 3.2.4. ***Equivalent norm.*** *For any operator $T \in B(H)$, $w(T) \leq \|T\| \leq 2w(T)$.*

Proof. If $\lambda = \langle Tx, x\rangle$ with $\|x\| = 1$, we have by Schwartz inequality

$$\begin{aligned} |\lambda| \leq & \ |\langle Tx, x\rangle| \\ \leq & \ \|Tx\|\|x\| \\ \leq & \ \|T\|\|x\|^2 \\ = & \ \|T\|. \end{aligned}$$

Clearly $w(T) \leq \|T\|$. To prove the other inequality, we use polarization identity

$$4\langle Tx, y\rangle = \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle.$$

Hence by direct computation we get

$$\begin{aligned} 4|\langle Tx, y\rangle| &\leq w(T)\left\{\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right\} \\ &= 4w(T)[\|x\|^2 + \|y\|^2]. \end{aligned}$$

Now choosing $\|x\| = \|y\| = 1$, we have $4\langle Tx, y\rangle \leq 4w(T)(2)$, and so $4\langle Tx, y\rangle \leq 8w(T)$. This implies that

$$\|T\| \leq 2w(T).$$

□

For details on the above result, see [16].

Theorem 3.2.4 implies that $T = 0$ whenever $w(T) = 0$. But we notice that this result is not valid in a real Hilbert space, as the example below shows.

Example 3.2.5. Let $H = \mathbb{R} \times \mathbb{R}$ and T the operator represented by the matrix

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $x = (x_1, x_2)$, $\|x\| = 1$, we have

$$Tx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

and therefore $Tx = (-x_2, x_1)$ and $\langle Tx, x \rangle = 0$. However, $\|T\| = 1$.

Now, we look at extreme cases of the inequality in Theorem 3.2.4. We recall that the spectral radius is given by $\gamma(T) = \sup\{|\lambda|,\ \lambda \in \sigma(T)\}$ and the point spectrum by $P\sigma(T) = \{\lambda \in \sigma(T),\ Tx = \lambda x \text{ for some } x \in H\}$.

Theorem 3.2.6. *If* $w(T) = \|T\|$, *then* $\gamma(T) = \|T\|$.

Proof. Let $w(T) = \|T\| = 1$. Then there is a sequence of unit vectors (x_n)

such that $\langle Tx_n, x_n \rangle \longrightarrow \lambda \in W(T), \ |\lambda| = 1$. That is

$$\begin{aligned} \langle Tx_n, x_n \rangle &\rightarrow \langle \lambda x_n, x_n \rangle \\ &= \lambda \langle x_n, x_n \rangle \\ &= \lambda \|x_n\|^2 \\ &= \lambda. \end{aligned}$$

From the inequality

$$|\langle Tx_n, x_n \rangle| \leq \|Tx_n\| \leq 1,$$

we have $\|Tx_n\| \longrightarrow 1$. Hence,

$$\|(T - \lambda I)x_n\|^2 = \|Tx_n\|^2 - \langle Tx_n, \lambda x_n \rangle - \langle \lambda x_n, Tx_n \rangle + \|\lambda_n\|^2 \longrightarrow 0.$$

Hence $\lambda \in \sigma_{app}(T)$ and $\gamma(T) = 1$. □

Theorem 3.2.7. *If $\lambda \in W(T)$, $|\lambda| = \|T\|$, then $\lambda \in P\sigma(T)$.*

Proof. Let $\lambda = \langle Tx, x \rangle$, $\|x\| = 1$. Then

$$\|T\| = |\lambda| = |\langle Tx, x \rangle| \leq \|Tx\| \leq \|T\|.$$

So $|\langle Tx, x \rangle| = \|Tx\|\|x\|$. Thus $Tx = \mu x$ for some $\mu \in \mathbb{C}$. However, $\lambda = \langle Tx, x \rangle = \langle \mu x, x \rangle = \mu$ and hence $Tx = \lambda x$. □

The above theorem 3.2.7 can be found in [16]. We now proceed to give our main results in this study in the next section.

3.3 Main results on the spectrum and numerical range.

Our aim in this section is to show that the $\sigma(T)$ is included in the $\overline{W(T)}$. It is sufficient to look at the boundary of the spectrum. We first give the following theorem.

Theorem 3.3.1. ***Theorem.***
The boundary of the spectrum $\partial\sigma(T)$ is contained in the approximate point spectrum $\sigma_{app}(T)$. That is $\partial\sigma(T) \subseteq \sigma_{app}(T)$. (Where ∂ denotes the boundary.)

Proof. We first prove a result. If T_n is a sequence of bounded invertible operators on H and $T_n \longrightarrow T$ in norm. That is $\lim_{n\longrightarrow\infty} \|T_n - T\| = 0$, where $T \in B(H)$ is not invertible, then $0 \in \sigma_{app}(T)$.
Indeed to see this, since T is not invertible, $T - 0I$ is not invertible, so $0 \in \sigma(T)$. But $\sigma(T) = \sigma_{app}(T) \cup \Gamma(T)$. Therefore, this implies that $0 \in \sigma_{app}(T)$ *or* $0 \in \Gamma(T)$. If we already have $0 \in \sigma_{app}(T)$, the proof is over. Otherwise $\mathbf{R}_T$ is not dense in H. Hence there is a nonzero $x \in H$ such that $x \perp \mathbf{R}_T$.
since $T_n's$ are invertible and hence bijections so $x_n = \frac{T_n^{-1}x}{\|T_n^{-1}x\|}$ is uniquely determined and $x_n \neq 0$. That is, $T_n^{-1}x_n \neq \overline{0}$. Hence,

$$\|\frac{T_n^{-1}x}{\|T_n^{-1}x\|}\| = 1, \;\; \forall \; n \in \mathbb{N}$$

Now ;

$$T_n x_n = T_n(\frac{T_n^{-1}x}{\|T_n^{-1}x\|}) = \frac{x}{\|T_n^{-1}x\|} \in \; \mathbf{R}_T^{\perp}$$

(Since $x \in \mathbf{R}_T^{\perp}$) therefore, $T_n x_n \in \mathbf{R}_T^{\perp}$, $\forall \ n \in \mathbb{N}$.

Now,

$$\|T_n x_n - T x_n\| \leq \|T_n - T\| \|x_n\| = \|T_n - T\| \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty (\text{by hypothesis}).$$

But $T_n \in \mathbf{R}_T$ obviously, $\forall \ n \in \mathbb{N}$. That is $T_n x_n \perp \mathbf{R}_T$ and $T x_n \in \mathbf{R}_T$. Therefore, $T_n x_n \perp T x_n, \ \ \forall \ \ n \in \mathbb{N}$, since, by pythagorean theorem, $\|x \pm iy\|^2 = \|x\|^2 + \|y\|^2$ for $x \perp y$ and $\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \|y\|^2 - \langle x, y \rangle - \langle y, x \rangle$. Therefore, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$ since $\langle x, y \rangle = 0$ for $x \perp y$.

Now it follows that

$$\|T_n x_n - T x_n\|^2 = \|T_n x_n\|^2 + \|T x_n\|^2. \ (\text{by Pythagorean theorem}).$$

But since, $\|T_n x_n - T x_n\|^2 \longrightarrow 0$, we have

$$\|T_n - T\| \longrightarrow 0, \ \ \texttt{implying} \ \ \|T x_n\| \longrightarrow 0, \ \ \texttt{as n} \ \ \longrightarrow \infty$$

That is,

$$\|(T - 0I) x_n\| \xrightarrow{s} 0$$

That is,

$$0 \in \sigma_{app}(T).$$

Let $\lambda \in \partial\sigma(T)$, $\Big($Note that $\sigma(T)$ is closed$\Big)$ then we can choose a sequence (λ_n) of points of $\rho(T)$ such that

$$\lambda_n \longrightarrow \lambda \ \text{as} \ \text{n} \longrightarrow \infty.$$

That is,

$$|\lambda_n - \lambda| \longrightarrow 0 \text{ as n} \longrightarrow \infty.$$

Now,

$$\begin{aligned}
\|(T - \lambda_n I) - (T - \lambda I)\| &= \|(\lambda - \lambda_n)I\| \\
&= |\lambda_n - \lambda|\|I\| \\
&= |\lambda_n - \lambda| \longrightarrow 0 \text{ as n} \longrightarrow \infty
\end{aligned}$$

But $(T - \lambda_n I)$ is invertible since $\lambda \in \rho(T)$ and $(T - \lambda I)$ is not invertible.

Therefore $0 \in \sigma_{app}(T - \lambda I)$ (by the result proved) that is, there exists a sequence $y_n \in H$ such that $\|y_n\| = 1$ and $\|(T - \lambda I)y_n\| \longrightarrow 0$ as n $\longrightarrow \infty$. That is, $\lambda \in \sigma_{app}(T)$. Therefore,

$$\partial\sigma(T) \subseteq \sigma_{app}(T).$$

□

Now, we proceed to establish the relationship between the spectrum and the numerical range in the following theorem which is a known result but with reference to the work of Bachman and Narici [4], we give a new approach to its proof;

Theorem 3.3.2. ***Theorem.***
Let H be a complex Hilbert space, $B(H)$ a set of bounded linear operators on H. Let $T \in B(H)$, then $\sigma(T) \subseteq \overline{W(T)}$ and $\|T\| \in \overline{W(T)}$ if and only if $\|T\| \in \sigma_{app}(T)$.

Proof. If $\lambda \notin \overline{W(T)}$, then $d = dist(\lambda, \overline{W(T)}) > 0$, (where dist is the distance function derived from the modulus in $\mathbb{C}$) then $\lambda I - T$ has an inverse and $\|(\lambda I - T)^{-1}\| < \frac{1}{d}$. So by definition of distance d, we have

$$d \leq |\langle Tx, x\rangle - \lambda|, \ \ \forall \ x \in H \ \ \|x\| = 1.$$

This implies that,

$$d\|x\|^2 \leq |\langle (T - \lambda I)x, x\rangle|, \ \ \forall \ x \in H$$

and using the Cauchy-Schwarz inequality, we see that

$$\|(T - \lambda I)x\| \geq d\|x\|.$$

Now, since $(T - \lambda I)$ is bounded from below, $(T - \lambda I)^{-1}$ exists on $\mathbf{R}_{(T-\lambda I)}$ and is bounded; moreover

$$\|(T - \lambda I)^{-1}y\| \geq d^{-1}\|y\|, \ \ \forall \ y \in \mathbf{R}_{(T-\lambda I)}.$$

Hence, there are only two possibilities, that is, $\lambda \in \rho(T)$ or $\lambda \in R\sigma(T)$ Suppose $\lambda \in R\sigma(T)$. Since,

$$\begin{aligned} \{\overline{\mathbf{R}_{(T-\lambda I)}}\}^{\perp} &= \{\mathbf{R}_{(T-\lambda I)}\}^{\perp} \\ &= ker(T^* - \overline{\lambda} I) \ \ (Nullspace) \end{aligned}$$

If $\lambda \in R\sigma(T)$, then $\{\overline{\mathbf{R}_{(T-\lambda I)}}\}^{\perp} \neq \{0\}$, that is, $ker(T^* - \overline{\lambda} I) \neq \{0\}$, and hence $\overline{\lambda}$ is an eigenvalue of T^*.

If $x \in H$, $\|x\| = 1$ and is such that $T^*x = \overline{\lambda}x$, then

$$Tx = \lambda x \;\; for \;\; x \neq 0$$

$$\begin{aligned}
\langle Tx, x \rangle &= \langle x, T^*x \rangle \\
&= \langle x, \overline{\lambda}x \rangle \\
&= \lambda \langle x, x \rangle \\
&= \lambda \|x\|^2 \\
&= \lambda
\end{aligned}$$

which implies that $\lambda \in W(T)$, a contradiction. Hence, if $\lambda \notin \overline{W(T)}$, then $\lambda \notin \sigma(T)$; this shows that

$$\sigma(T) \subseteq \overline{W(T)}.$$

So from $\|(T-\lambda I)^{-1}y\| \geq d^{-1}\|y\|$, we have $\|(T-\lambda I)^{-1}\| \leq d^{-1}$. Now on the other hand, $P\sigma(T) \subset W(T)$ and $\sigma_{app}(T) \subset \overline{W(T)}$ such that $|\lambda| = \|T\|$. To see this, if $\lambda \in P\sigma(T)$, then there exists $x \in H$ such that $\|x\| = 1$ and $Tx = \lambda x$. Then,

$$\begin{aligned}
\langle Tx, x \rangle &= \langle \lambda x, x \rangle \\
&= \lambda \langle x, x \rangle \\
&= \lambda \|x\|^2 \\
&= \lambda
\end{aligned}$$

Thus $\lambda \in W(T)$.

Now since $\sigma_{app}(T) \subset \sigma(T)$ and $\sigma(T) \subset \overline{W(T)}$, we have $\sigma_{app}(T) \subset \overline{W(T)}$.

Alternatively, $\lambda \in W(T)$ implies that there exists a sequence (x_n) of unit vectors in H such that

$$\lim_{n \longrightarrow \infty} \|(\lambda I - T)x_n\| = 0.$$

Since for such x_n

$$\begin{aligned} |\lambda - \langle Tx_n, x_n \rangle| &= |\langle (\lambda I - T)x_n, x_n \rangle| \\ &\leq \|(\lambda I - T)x_n\| \|x_n\| \\ &\leq \|(\lambda I - T)x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Thus

$$\lambda = \lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle.$$

Therefore, it follows that $\lambda \in \overline{W(T)}$.

Since $|\lambda| = \|T\| = w(T) = \sup\{|\lambda| \ : \ \lambda \in \sigma(T)\}$. So $\|T\| \in \sigma_{app}(T)$ implies that $\|T\| \in \overline{W(T)}$. □

Example 3.3.3. Consider the Hilbert space $\mathbb{C}^2$ of dimension two over $\mathbb{C}$ and take the orthonormal basis $\{e_1, e_2\}$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Define $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ linearly through $Te_1 = e_2$ and $Te_2 = \bar{0}$. Thus matrix of T with respect to the given orthonormal basis is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

0 is the only eigenvalue of T, thus $\sigma(T) = P\sigma(T) = \{0\}$, since $\mathbb{C}^2$ is finite

dimensional. Let $x = (z_1, z_2)$ such that $z_1, z_2 \in \mathbb{C}$; so $x = z_1 e_1 + z_2 e_2$

$$\begin{aligned} Tx &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ z_1 \end{bmatrix} \\ &= 0e_1 + z_1 e_2 \\ &= z_1 e_2 \\ &= (0, z_1). \end{aligned}$$

Consequently,

$$\begin{aligned} \langle Tx, x \rangle &= \langle (0, z_1), (z_1, z_2) \rangle \\ &= \begin{bmatrix} 0 & z_1 \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} \\ &= 0\overline{z_1} + z_1 \overline{z_2} \\ &= z_1 \overline{z_2} \end{aligned}$$

If $\|x\| = 1$, then $|z_1|^2 + |z_2|^2 = 1$. Thus

$$W(T) = \{ z_1 \overline{z_2} \ : \ z_1, z_2 \in \mathbb{C} \text{ and } |z_1|^2 + |z_2|^2 = 1. \}$$

Now let $\lambda = z_1 \overline{z_2}$, so we have $\lambda = |z_1||\overline{z_2}| = |z_1| \sqrt{1 - |z_1|^2}$; hence

$$W(T) = \{ \lambda \in \mathbb{C} \ : \ |\lambda|^2 = |z_1|^2 (1 - |z_1|^2) \text{ where } 0 \leq |z_1| \leq 1 \text{ and } z_1 \in \mathbb{C} \}$$

If $|z_1| = 0$; or 1, then $\lambda = 0$.

We find the maximum value of λ as $|z_1|$ varies over the closed interval $[0, 1]$. We can use the technique of calculus or the following procedure

$$|\lambda|^2 = |z_1|^2(1 - |z_1|^2) = (|z_1|^2 - |z_1|^4) = (\frac{1}{4} - (|z_1| - \frac{1}{2})^2)^2.$$

Since $|\lambda| \geq 0$, we note that maximum value of $|\lambda|$ is $\frac{1}{2}$ and occurs when $|z_1| = \frac{1}{2}$. Hence

$$W(T) = \left\{\lambda \in \mathbb{C} \ : \ |\lambda| \leq \frac{1}{2}\right\}.$$

Also $w(T) = \frac{1}{2}$ as is seen from the set just above.

Alternatively, we may also observe that

$$|\langle Tx, x\rangle| = |z_1\overline{z_2}| \leq \frac{1}{2}(|z_1|^2 + |z_2|^2) = \frac{1}{2}.$$

For $z_1 = z_2 = \frac{1}{\sqrt{2}}$, we obtain $w(T) \geq (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$. Hence $w(T) = \frac{1}{2}$. Note that

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| \ : \ x \in H \ \text{and} \ \|x\| = 1\} \ \text{for} \ x = (z_1, z_2) \\ &= \sup\{\|(0, z_1)\| \ : \ x = (z_1, z_2) \ \text{and} \ \|x\| = 1\} = 1. \end{aligned}$$

Thus $w(T) = \frac{1}{2}\|T\|$ for this operator.

3.4 Normal operators.

In this section, we consider a normal operator and investigate the relationship between its spectrum and numerical range. We actually establish this using the spectral and the numerical radii. We first look at basic examples of normal operators.

3.4.1 Examples of normal operators

Example 3.4.1. All self-adjoint operators are normal.

Proof. If T is self-adjoint, then $T = T^*$. Then for all $x \in H$,

$$\begin{aligned} \|T^*Tx\|^2 &= \langle T^*Tx, T^*Tx\rangle \\ &= \langle TTx, TTx\rangle \\ &= \langle TT^*x, TT^*x\rangle \\ &= \|TT^*x\|^2 \\ \implies T^*T &= TT^*. \end{aligned}$$

□

Example 3.4.2. All unitary operators are normal.

Proof. The proof of this follows from the definition 1.1.21 of a unitary operator. □

3.4.2 Further properties of normal operators and spectrum

For normal operators $T \in B(H)$, we show the following results:

Theorem 3.4.3. *Let $T \in B(H)$ be normal, then T^* is also normal.*

Proof. If T is normal it implies that

$$\begin{aligned} T^*T &= TT^* \\ T^*T^{**} &= T^{**}T^* \\ (TT^*)^* &= (T^*T)^*. \end{aligned}$$

Thus $TT^* = T^*T$. Which implies that T^* normal. □

Theorem 3.4.4. *If $T \in B(H)$ is normal, then the spectral radius $\gamma(T)$ equals $\|T\|$. That is $\gamma(T) = \|T\|$.*

Proof. For all $T \in B(H)$,

$$\begin{aligned} \|T^*T\| &= \sup\{\|T^*Tx\| : x \in H, \ \|x\| = 1\} \\ &\leq \sup\{\|T\|^2\|x\|^2 : x \in H, \ \|x\| \leq 1\} \\ &= \|T\|^2. \end{aligned}$$

To establish the reverse inequality, we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &= |\langle T^*Tx, x\rangle| \text{ (since } T^*T \geq 0) \\ &\leq \|T^*Tx\|\|x\|. \\ &\leq \|T^*T\|\|x\|^2. \end{aligned}$$

Thus $\|Tx\| \leq \sqrt{\|T^*T\|}\|x\| \ \forall x \in H$. That is $\|T\| \leq \sqrt{\|T^*T\|}$, implying that $\|T\|^2 \leq \|T^*T\|$, which is the reverse inequality. Therefore, $\|T^2\| =$

$\|T\|^2$. By induction we obtain that for self-adjoint T,

$$\|T^{2^n}\| = \|T\|^{2^n}, \quad \forall \; n \in \mathbb{N}.$$

Now, let T be normal, since $\gamma(T) \leq \|T\|$ always hold, we only have to prove that $\gamma(T) \geq \|T\|$. Since $\gamma(T) = \gamma(T^*)$, we have

$$\begin{aligned}
(\gamma(T))^2 &= \gamma(T)\gamma(T) \\
&= \lim_{n \longrightarrow \infty} \{\|T^{2^n}\| \|(T^*)^{2^n}\|^{\frac{1}{2^n}}\} \\
&= \lim_{n \longrightarrow \infty} \|T^{2^n}(T^*)^{2^n}\|^{\frac{1}{2^n}} \\
&= \lim_{n \longrightarrow \infty} \|(TT^*)^{2^n}\|^{\frac{1}{2^n}} \\
&= \|TT^*\| \\
&= \|T\|^2.
\end{aligned}$$

So this implies that $\gamma(T) = \|T\|$. □

Theorem 3.4.5. *Let $T \in B(H)$ be normal, then T is normal if and only if $\|Tx\| = \|T^*x\|, \; \forall \; x \in H$.*

Proof. We first assume that T is normal. Then,

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx \rangle \\
&= \langle x, T^*Tx \rangle \\
&= \langle x, TT^*x \rangle \\
&= \langle T^*x, T^*x \rangle \\
&= \|T^*x\|^2 \\
\Longrightarrow \quad \|Tx\| &= \|T^*x\|.
\end{aligned}$$

Conversely, we assume that $T^*T = TT^*, \;\; \forall \;\; x \in H$ and prove that T is normal.

Now,

$$\begin{aligned}
\langle T^*Tx, x\rangle &= \langle Tx, Tx\rangle \\
&= \|Tx\|^2 \\
&= \|T^*x\|^2 \\
&= \langle T^*x, T^*x\rangle \\
&= \langle TT^*x, x\rangle \\
\Longrightarrow \;\; T^*T &= TT^* \;\; \Rightarrow \;\; T \text{ is normal.}
\end{aligned}$$

□

We recall that normal operators, those T for which $T^*T = TT^*$, may be regarded as a generalization of self-adjoint operators T in which T^* need not be exactly T but commutes with T.

Now we state and prove the following theorem,

Theorem 3.4.6. *If T is normal, then $\|T^n\| = \|T\|^n$, $n = 1, 2, ...$ Moreover, $\gamma(T) = w(T) = \|T\|$.*

Proof. For any $x \in H$,

$$\begin{aligned}
\|Tx\|^2 &= \langle T^*Tx, x\rangle \\
&\leq \|T^*Tx\| \\
\text{Hence} \;\; \|T\|^2 &\leq \|T^2\|.
\end{aligned}$$

Conversely,

$$\begin{aligned} \|T^2\| &= \|TT\| \\ &\leq \|T\|\|T\| \\ \text{Hence } \|T^2\| &\leq \|T\|^2. \end{aligned}$$

Therefore, since $\|T^2\| \leq \|T\|^2$ and $\|T\|^2 \leq \|T^2\|$ we conclude that $\|T^2\| = \|T\|^2$. Now, for any $x \in H$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|T^*T^n x\|^2 &= \langle T^*T^n x, T^*T^n x\rangle \\ &= \langle TT^*(T^n x), T^n x\rangle \end{aligned}$$

Since T is normal, we have $T^*T = TT^*$. Therefore,

$$\begin{aligned} \|T^*T^n x\|^2 &= \langle T^*T(T^n x), T^n x\rangle \\ &= \langle T^*T^{n+1}x, T^n x\rangle \\ &= \langle T^{n+1}x, T^{n+1}x\rangle \\ &= \|T^{n+1}x\|^2. \end{aligned}$$

$$\text{That is, } \|T^*T^n x\| = \|T^{n+1}x\|. \tag{3.4.1}$$

Now,

$$\begin{aligned}
\|T^n x\|^2 &= \langle T^n x, T^n x\rangle \\
&= \langle T^* T^n x, T^{n-1} x\rangle \\
&\leq \|T^* T^n x\| \|T^{n-1} x\| \\
&\leq \|T^{n+1} x\| \|T^{n-1} x\| \\
&\leq \|T^{n+1}\| \|T^{n-1}\| \|x\|^2.
\end{aligned}$$

Taking sup on both sides with $\|x\| = 1$ we obtain,

$$\|T^n\|^2 \leq \|T^{n+1}\| \|T^{n-1}\|, \ \forall \ n \in \mathbb{N}.$$

Suppose $\|T^k\| = \|T\|^k$ for $1 \leq k \leq n$, then we show that it is true for $k = n+1$. Therefore,

$$\begin{aligned}
\|T\|^{2n} &= (\|T\|^n)^2 \\
&= \|T^n\|^2 \text{ (by induction)} \\
&\leq \|T^{n+1}\| \|T^{n-1}\| \\
&= \|T^{n+1}\| \|T\|^{n-1} \text{ (by induction)}
\end{aligned}$$

$$\text{Therefore, } \|T\|^{2n} \leq \|T^{n+1}\| \|T\|^{n-1}. \tag{3.4.2}$$

Now dividing equation 3.4.2 both sides by $\|T\|^{n-1}$, we get

$$\begin{aligned}
\|T\|^{2n}(\|T\|^{n-1})^{-1} &\leq \|T^{n+1}\| \\
\|T\|^{2n}\|T\|^{1-n} &\leq \|T^{n+1}\| \\
\|T\|^{2n+1-n} &\leq \|T^{n+1}\| \\
\|T\|^{n+1} &\leq \|T^{n+1}\|.
\end{aligned}$$

$$\text{That is,} \quad \|T\|^{n+1} \leq \|T^{n+1}\| \tag{3.4.3}$$

On the other hand,

$$\begin{aligned}
\|T^{n+1}\| &= \|\underbrace{T.T.T...T}_{n+1 \text{ times}}\| \\
&\leq \underbrace{\|T\|\|T\|\|T\|...\|T\|}_{n+1 \text{ times}} \\
&= \|T\|^{n+1}.
\end{aligned}$$

So that,

$$\|T^{n+1}\| \leq \|T\|^{n+1}. \tag{3.4.4}$$

From equations (3.4.3) and (3.4.4), we get $\|T^{n+1}\| = \|T\|^{n+1}$.

Thus, $\|T^n\| = \|T\|^n$, $\forall\ n$ and for T normal. Moreover,

$$\begin{aligned}
\gamma(T) &= \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} \\
&= \lim_{n \longrightarrow \infty} (\|T\|^n)^{\frac{1}{n}} \\
\text{Hence} \quad \gamma(T) &= \|T\|.
\end{aligned}$$

Now by theorem 3.2.6, we conclude that $\gamma(T) = w(T) = \|T\|$. □

3.5 Algebra numerical range.

Definition 3.5.1. Let A be a complex normed algebra with unit. Denote by $E(A)$ the set of states on A. The algebra numerical range of an element $T \in A$ is defined by

$$V(T) = \{f(T) : f \in E(A)\}. \quad (3.5.1)$$

It is well-known that $V(T)$, is a compact convex subset of the complex plane. See [5].

3.5.1 Properties of algebra numerical range.

We note that from now on, $B(H)$ is considered as an algebra of bounded linear operators on a Hilbert space H as opposed to the previous considerations as a set. Algebra numerical range $V(T)$ has the following properties:

Theorem 3.5.2. *For all $T, S \in B(H)$*

(i) $V(T)$ is non-empty compact convex subset of scalars.

(ii) $V(\lambda I + \mu T) = \lambda + \mu V(T)$ for I is the identity in $B(H)$ and $\lambda, \mu \in \boldsymbol{K}$.

(iii) $V(T + S) = V(T) + V(S)$.

(iv) $|\lambda| \leq \|T\|$, for all $\lambda \in V(T)$.

Proof.

(i) Let $T : H \longrightarrow H$, then for all $T \in B(H)$, we show that the set

$$V(T) = \{f(T) : f(I) = 1 = \|f\|\} \text{ is convex.}$$

Let $\lambda_1,\ \lambda_2 \in V(T)$. We seek to show that $\alpha\lambda_1 + (1-\alpha)\lambda_2 \in V(T)$ for $0 < \alpha \leq 1$.

Now this implies that, there exists functionals $\phi_1,\ \ \phi_2 \in E(B(H))$ such that

$$\phi_1(T) = \lambda_1,\ \phi_2(T) = \lambda_2$$

and

$$\phi_1(I) = 1 = \|\phi_1\|,\ \ and\ \ \phi_2(I) = 1 = \|\phi_2\|$$

define ϕ by $\phi(T) = \alpha\phi_1(T) + (1-\alpha)\phi_2(T)$.

Then for $0 < \alpha \leq 1$ and $\beta_1,\ \beta_2 \in \mathbf{K}$,

$$\begin{aligned}
\phi(\beta_1 T_1 + \beta_2 T_2) &= \alpha\phi_1(\beta_1 T_1 + \beta_2 T_2) + (1-\alpha)\phi_2(\beta_1 T_1 + \beta_2 T_2) \\
&= \alpha\phi_1(\beta_1 T_1) + \alpha\phi_1(\beta_2 T_2) + (1-\alpha)\phi_2(\beta_1 T_1) + (1-\alpha)\phi_2(\beta_2 T_2) \\
&= \alpha\beta_1\phi_1(T_1) + \alpha\beta_2\phi_1(T_2) + (1-\alpha)\beta_1\phi_2(T_1) + (1-\alpha)\beta_2\phi_2(T_2) \\
&= \beta_1\{\alpha\phi_1(T_1) + (1-\alpha)\phi_2(T_1)\} + \beta_2\{\alpha\phi_1(T_2) + (1-\alpha)\phi_2(T_2)\} \\
\phi(\beta_1 T_1 + \beta_2 T_2) &= \beta_1\phi(T_1) + \beta_2\phi(T_2).
\end{aligned}$$

Hence ϕ is linear.

Next, we show that $\|\phi\| = 1$.

Since, $\phi(I) = \alpha\phi_1(I) + (1-\alpha)\phi_2(I) = 1$.

Now, it follows that, $1 = |\phi(I)| \leq \|\phi\|\|I\| = \|\phi\|$.

$$\begin{aligned} |\phi(T)| &= |\alpha\phi_1(T) + (1-\alpha)\phi_2(T)| \\ &\leq \alpha|\phi_1(T)| + (1-\alpha)|\phi_2(T)| \\ &\leq \alpha\|\phi_1\|\|T\| + (1-\alpha)\|\phi_2\|\|T\| \\ &\leq \alpha\|T\| + (1-\alpha)\|T\| \\ &= \|T\|. \end{aligned}$$

Thus, $\|\phi\| \leq 1$, $\|\phi\| \geq 1$ so $\|\phi\| = 1$. We note that the norm of ϕ is given by

$$\|\phi\| = \sup\{|\phi(T)| : \|T\| \leq 1\}.$$

It follows that $\phi(T) \in V(T)$. Hence $V(T)$ is convex.
For compactness and non-emptiness of $V(T)$, we refer to H. M. Sadia [17].

(ii) For all $\lambda, \mu \in \mathbf{K}$,

$$\begin{aligned} V(\lambda I + \mu T) &= \{f(\lambda I + \mu T) : f \in E(B(H))\} \\ &= \{\lambda f(I) + \mu f(T) : f \in E(B(H))\} \\ &= \{\lambda + \mu f(T) : f \in E(B(H))\} \\ &= \lambda + \mu V(T). \end{aligned}$$

(iii)

$$\begin{aligned} V(T+S) &= \{f(T+S) : f \in E(B(H))\} \\ &= \{f(T) : f \in E(B(H))\} + \{f(S) : f \in E(B(H))\} \\ &= V(T) + V(S). \end{aligned}$$

(iv)$|\lambda| \leq \|T\|$, for all $\lambda \in V(T)$. If $\lambda \in V(T)$, then $\lambda = f(T)$ for all $f \in E(A)\}$. Then

$$|\lambda| = |f(T)| \leq \|f\|\|T\| = \|T\| \text{ since } \|f\| = 1.$$

Hence, $|\lambda| \leq \|T\|$, for all $\lambda \in V(T)$. □

Chapter 4

SUMMARY AND RECOMMENDATION

In this last chapter, we draw conclusions and make recommendations based on our objective of study and the results obtained.

4.1 Summary

In the conclusion of our research, we would like to give a summary of our study. In chapter one, we discussed the background information, basic concepts, definitions, notations and symbols that pertains to this study. Chapter two, dealt with numerical ranges and discussed exhaustively its properties, for instance convexity, closedness among others. We further considered some results on the numerical range.

In chapter three, we defined the spectrum of a bounded linear operator on Hilbert space and gave its properties. We further established that, the spectrum of a bounded linear operator is contained in the closure of its numerical range. Moreover, we looked at normal operators and its examples where we established its relationship with the spectrum. Lastly in the same chapter, we included some basic properties of algebra numerical range.

4.2 Recommendation.

From this study, we recommend that the relationship between the spectra and numerical ranges can still be investigated for other operators such as hyponormal operators, subnormal, quasinormal, paranormal operators among other large classes of normal operators. Further, the relationship between algebra numerical ranges and the spectra can also be explored. Much attention can be directed towards these mentioned areas.

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