The Almost Holomophic Functional Calculus

by

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Abstract

A closed densely defined operator H, on a Banach space $\mathcal{X}$, whose spectrum is contained in $\mathbb{R}$ and satisfies

$$\left\|(z-H)^{-1}\right\| \leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}} \qquad \forall\, z \notin \mathbb{R} \tag{1}$$

for some α , $\beta \geq 0$; $c > 0$, is said to be of $(\alpha,\ \beta) - type\ \mathbb{R}$. If instead of (1) we have

$$\left\|(z-H)^{-1}\right\| \leq c\frac{|z|^{\alpha}}{|\Im z|^{\beta}} \qquad \forall\, z \notin \mathbb{R}, \tag{2}$$

then H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$.

Examples of such operators include self-adjoint operators, Laplacian on $L^1(\mathbb{R})$, Schrödinger operators on $L^p(\mathbb{R}^n)$ and operators whose spectra lie in $\mathbb{R}$ and permit some control on $\left\|e^{iHt}\right\|$.

Important properties, especially those concerning the spectra, of operators of $(\alpha,\ \beta) - type\ \mathbb{R}$, are studied. These include some perturbation results.

For $\beta \in \mathbb{R}$, $\mathfrak{S}^{\beta}$ is the space of smooth functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $\left|f^{(r)}(x)\right| := \left|\frac{d^r f}{dx^r}\right| = O(\langle x\rangle^{\beta - r}),\ as\ |x| \rightarrow \infty\ \forall\, r \geq 0$. The space $\mathfrak{A} := \cup_{\beta<0}\mathfrak{S}^{\beta}$ is then a topological algebra under pointwise multiplication, containing the sub-algebra $C_c^{\infty}(\mathbb{R})$ of all smooth functions of compact support. The completions $\mathfrak{A}_n$ of $\mathfrak{A}$ or $C_c^{\infty}(\mathbb{R})$ with respect to the norms $\|f\|_n := \sum_{r=0}^{n}\int_{-\infty}^{\infty}\left|f^{(r)}(x)\right|\langle x\rangle^{r-1}\,dx$, are also algebras under pointwise multiplication.

Functions in $\mathfrak{A}$ include exponential function and spectral map

$r_w(x) := \frac{1}{w-x}$ for some $w \in \mathbb{C} \setminus \mathbb{R}$.

Given $f \in \mathfrak{A}$ and $n \geq 0$, the **almost analytic extension** of f to $\mathbb{C}$ is $\tilde{f}(x,y) := \left(\sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\right) \varphi(x,y)$, where $\varphi(x,y) := \tau\left(\frac{y}{\langle x\rangle}\right)$ and $\tau \in C^\infty$ with $\tau(s) = 1$ if $|s| \leq 1$ and $\tau = 0$ if $|s| \geq 2$.

If $n > \alpha > 0$, $f \in \mathfrak{A}$ and H is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$, then the integral

$$f(H) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z-H)^{-1} dx dy \tag{3}$$

is norm convergent and defines an operator in $\mathfrak{B}(\mathcal{X})$ with $\|f(H)\| \leq c \, \|f\|_{n+1}$. Then the map $f \mapsto f(H)$, is a functional calculus for H.

Using the constructed functional calculus, we define roots and exponentials of an $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ operator with $\sigma(H) \subseteq [\mu, \infty)$ for some appropriate μ. These allow us to treat various Cauchy's problems.

Chapter 1

Introduction

1.1 Basic concepts

In mathematics, a functional calculus is a theory allowing one to apply mathematical functions to mathematical operators. If f is a function, say a numerical function of a real number, and M is an operator, there is no particular reason why the expression

$$f(M)$$

should make sense. If it does, then we are not using f on its original function domain any longer. This passes nearly unnoticed if we talk about 'squaring a matrix', though, which is the case of $f(x) = x^2$ and M an $n \times n$ matrix. The idea of a functional calculus is to create a principled approach to this kind of overloading of the notation.

The most immediate case is to apply polynomial functions to a square matrix, extending what has just been discussed. In the finite dimensional case, the polynomial functional calculus yields quite a bit of information about the operator. For example, consider the family of polynomials which annihilates an operator T. This family is an ideal in the ring of polynomials. Furthermore, by the Cayley-Hamilton theorem, it is non-trivial. Since the ring of polynomials is a principal ideal domain, the ideal is generated by some polynomial m. m is precisely the minimal polynomial of T, and it can be used to calculate, for example, the exponential of T efficiently. The polynomial calculus is not as informative in the infinite dimensional case. Consider the unilateral shift with the polynomials calculus; the ideal defined above is now trivial. Thus one is interested in functional calculi more general than polynomials. The subject is closely linked to spectral theory, since for a diagonal matrix or multiplication operator, it is rather clear what the definitions should be.

In studying unbounded linear operators, it is desirable to gain as much information as possible by looking at the spectrum. One of the most powerful tools for such a study, if available, is a functional calculus. Perhaps the most well known functional calculus is Riesz–Dunford functional calculus,

$$f(H) := \frac{1}{2\pi i}\int_\Gamma f(z)(z-H)^{-1}dz. \tag{1.1}$$

Where $H \in \mathfrak{B}(\mathcal{X})$, f is holomorphic in an open neighbourhood containing $\sigma(H)$ and Γ surrounds $\sigma(H)$. To admit a richer functional calculus, H would be expected to satisfy more additional properties.

Of more interest in applications are the unbounded operators such as differential operators. Two problems immediately arise when one tries to extend (1.1) to unbounded operators. The spectrum, $\sigma(H)$ may be unbounded and functions holomorphic on $\sigma(H)$ may be unbounded.

Since $\int_\Gamma \|(z-H)^{-1}\|\,|dz|$ is no longer finite, it is not surprising that, even for bounded holomorphic f, $f(H)$ may not be bounded, even when $\|(z-H)^{-1}\|$ is $O(1+|z|)^{-1})$ as $|z| \to \infty$ [1], as is the case with a bounded operator. For example let D be the open disc in $\mathbb{C}$, let $iH := d/d\theta$ on $L^1(\partial D)$, the generator of the rotation group on ∂D and let $f := \chi_{[0,\infty)}$, the **characteristic function of** $[0,\infty)$. Then it is well known that $f(H)$, the projection of $L^1(\partial D)^2$ onto $H^1(D)$ is unbounded. Even on a Hilbert space, $f(H)$ may be unbounded, although f is bounded and holomorphic in an open neighbourhood containing the spectrum, $\sigma(H)$ (See [Mac89]).

For polynomially bounded f, one could modify (1.1) by replacing $(z-H)^{-1}xdz$ with $(z-H)^{-1}(r-z)^n x(r-z)^{-n}dz$ for n sufficiently large, x in the domain of H^n, $r \in \rho(H)$ (see [BD91, Mac86]). However, this collection of functions does not include functions of most interest (particularly in considering Abstract Cauchy problems) such as exponentials and cosines.

When H is unbounded there are a number of ways of bringing bounded operators into the picture and using their functional calculi to define a functional calculus for H. One could apply (1.1) to $(w-H)^{-1}$, for some $w \in \rho(H)$. This defines $f(H)$ only for f holomorphic in the neighbourhood of ∞ (see [DS58]), hence even more restrictive than the requirement of polynomial growth.

For unbounded operators with real spectra and slowly increasing resolvents when approaching the spectrum, we define $f(H)$ by modifying (1.1)

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[1] We shall write "$g(z)$ is $O(f(z))$ as $z \to \lambda$" to mean $\lim_{z\to\lambda} \frac{g(z)}{f(z)} < \infty$

[2] $L^p(\Omega)$ is the set of $p-$integrable functions, while $H^p(\Omega)$ are $p-$differentiable, bounded functions.

in such a way that f is replaced with $\frac{\partial \tilde{f}}{\partial \bar{z}}$, where $\tilde{f}$ is a sort of holomorphic extension of f initially defined and **smooth** (i.e. infinitely differentiable) on $\mathbb{R}$, and evaluate the integral on $\mathbb{C}$.

1.2 Literature Review

The existence of $(\alpha,\ \alpha+1)'-type\ \mathbb{R}$ operators can be traced in literature as sampled in this paragraph. According to H. Tanabe [Tan79], a closed operator T on $\mathcal{X}$ is said to be of *type* ω where $0 \leq \omega < \pi$, if supp $(T) \subseteq \overline{S_\omega}$ and for $0 < \epsilon < \pi - \omega$, $(S_\omega := \{z \ : \ |\arg(z)| < \omega\})$ there exists a positive constant c_ϵ such that $\|(z-T)^{-1}\| \leq \frac{c_\epsilon}{|z|}$, $z \notin \overline{S_{\omega+\epsilon}}$. So operators of *type 0* are a bit restrictive but correspond to $(0,\ 1)-type\ \mathbb{R}$ operators in our nomenclature. In both cases, the names indicate the location of the spectrum and the growth condition of the resolvent. DeLaubenfels [DeL93] also uses this idea in the study of $\alpha - type\ V$ operators, whose resolvents are polynomially ("degree α") bounded with spectra in V. Hille and Phillips [HP81] have shown that $\sigma(H_0) \subseteq [0,\infty)$ and $-H_0$ is the infinitesimal generator of Gauss-Weierstrass semigroup

$$(G_t f)x = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x-y)e^{-\frac{y^2}{4t}}dy,\ f \in L^p(\mathbb{R})$$

In Kato [Kat80, IX, sec. 18] it is shown that

$$\|(z-H_0)^{-1}\| \leq \frac{1}{|z|}\left|\sin^2 \frac{1}{2}\arg(z)\right|, \quad z \in \rho(H_0) = \mathbb{C} \setminus [0,\infty),$$

Theorem 8 in S. Nakamura [Nak94] is stated for $(\alpha,\ \alpha+1) - type\ \mathbb{R}$

operators. However, the proof given shows that Schrödinger operators are of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ which is stronger than the stated result.

For unbounded self-adjoint operators acting on a Hilbert space, Helffer and Sjöstrand [HS89] proved that the integral formula,

$$f(H) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z-H)^{-1}dxdy \tag{1.2}$$

is an alternative characterisation of the standard C_0-functional calculus. Contrary to their approach, our approach will not assume the existence of a functional calculus but **constructed** one in a more general Banach space setting. We then need to verify that our functional calculus, defined on a Banach space, coincides with C_0-functional calculus for an unbounded operator acting on a Hilbert space.

W. J. Ricker [Ric88, Theorem 1], showed that the Laplacian H_0, acting on $L^p, 1 < p < \infty\ p \neq 2$ is not scaler. We therefore conjecture that $\|f(H)\| > \|f\|_\infty$ for some $f \in \mathfrak{A}$.

For a class of operators associated with spectral distributions, Jazar [Jaz95] showed that if an integer $n \geq 1$, real $t > 0$ and $f(x)$ and $g(x)$ are two smooth functions which equal $e^{-x^n t}$ for $x \geq 0$ and $f(x),\ g(x) \to 0$ as $x \to -\infty$, and H is an operator of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ for some $\alpha > 0$ then $f(H) = g(H) =: e^{-H^n t}$ and

$$e^{-H^n(t_1+t_2)} = e^{-H^n t_1} e^{-H^n t_2}$$

for all $t_1, t_2 > 0$. Moreover there exists $c_n < \infty$ such that

$$\left\|e^{-H^n t}\right\| \leq c_n \tag{1.3}$$

for all $n \geq 1$ and $0 < t \leq 1$. We wish to proved it here within the context of our functional calculus.

1.3 Statement of the problem

First, we *fully characterise* the operators with real spectra and slowly increasing resolvents when approaching the spectrum. We shall *identify important operators* that arise from familiar physical problems, and *study their connection with this class.* We shall study properties of these operators and attempt to *obtain some perturbation results.*

Secondly, we study the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$. Among other things, we wish to *establish that $C_c^\infty(\mathbb{R})$[3] is dense in this algebra.* We shall characterise this algebra fully. It would be of great significance from application point of view to *demonstrate that important functions like $x \mapsto e^x$ are either in the algebra or can be extended to functions in the algebra.*

Thirdly tools developed above are used to *define the almost analytic functional calculus for operators with real spectra and slowly growing resolvents when approaching the spectrum.* An immediate application of the

[3] $C^p(\mathbb{R})$ are $p-$differentiable complex valued functions on $\mathbb{R}$, while $C^p(\mathbb{R})$ are $p-$differentiable functions on $\mathbb{R}$ with compact support

functional calculus would be in the proof of standard theorems like the spectral mapping theorem.

1.4 Objectives of the study

- To fully characterize the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$.
- To define the almost analytic functional calculus for operators with real spectra and slowly growing resolvents when approaching the spectrum.
- To provide alternative, easier proofs to known results using the functional calculus.
- To tackle outstanding problems whose solutions need the existence of the right functional calculus.

1.5 Significance of the study

Using the constructed functional calculus, we wish to define roots and exponentials of an $(\alpha,\ \alpha+1) - \textit{type}\ \mathbb{R}$ operator with $\sigma(H) \subseteq [\mu, \infty)$ for some appropriate μ. These allow us to treat various Cauchy's problems. We examine applications for our functional calculus in semigroup theory, and some aspects of L^p spaces.

1.6 Research Methodology

For completeness of our exposition, we will often re-state known results. However, for the most part we will omit the proofs. Instead, we will indicate where the proofs may be found. In some isolated cases, which will be explicitly indicated, alternative proofs to the known results will be provided by taking advantage of the functional calculus constructed here. In so doing the effectiveness of the functional calculus will be demonstrate in greatly simplifying existing proofs. On the other hand, a lot of technical details will be relegated to the appendices so as to facilitate a free flow of the presentation. For this reason, the appendices will be considered to be integral parts of the study.

1.7 Notations and organization of the study

Let $\mathcal{X}$,$\mathcal{Y}$ denote Banach spaces. Unless otherwise stated we will assume that the underlying field is the complex field $\mathbb{C}$[4] and the norm on $\mathcal{X}$ is $\|.\|$. Except in specific cases, which will be stated, we shall denote arbitrary complex numbers by $z := x + iy$ or $w := u + iv$, $x, y, u, v \in \mathbb{R}$ and $i := \sqrt{-1}$. Let $\mathbb{R}^+ := [0, \infty)$ be the non-negative real numbers, $\mathbb{N} := \{1, 2, \ldots\}$ be the positive integers, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ be the non-negative integers. $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ will denote the space of all bounded linear operators from the linear space $\mathcal{X}$ to linear space $\mathcal{Y}$. We write $\mathfrak{B}(\mathcal{X}) := \mathfrak{B}(\mathcal{X}, \mathcal{X})$. "$H$ is an operator on $\mathcal{X}$" means H is a linear operator from its **domain**, $\mathfrak{D}(H) \subseteq \mathcal{X}$ to $\mathcal{X}$. We will denote the **range** of H by $\mathfrak{R}(H)$.

Definition 1.7.1

An operator H on $\mathcal{X}$ is said to be **closed** if its graph

$$\mathcal{G}(H) := \{(f, Hf) \ : \ f \in \mathfrak{D}(H)\}$$

is a closed subspace of $\mathcal{X} \times \mathcal{X}$ or equivalently, $f_n \in \mathfrak{D}(H)$, $f_n \to f$ and $Hf_n \to g$ implies $f \in \mathfrak{D}(H)$ and $Hf = g$.

Definition 1.7.2

An operator H on $\mathcal{X}$ is said to be **densely defined** if its domain $\mathfrak{D}(H)$ is dense in $\mathcal{X}$.

[4] $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ denote complex, real, and rational number fields respectively, while $\mathbb{Z}$ and $\mathbb{N}$ denote integers and positive integers respectively.

Let H be an operator on $\mathcal{X}$. The **resolvent set** of H is

$$\rho(H) := \{z \in \mathbb{C} \ : \ zI - H : \mathfrak{D}(H) \to \mathcal{X} \text{ is bijective and } (zI-H)^{-1} \in \mathfrak{B}(\mathcal{X})\}.$$

Observe that if $\rho(H) \neq \emptyset$ then H is closed. See [Dav95, Lemma 1.1.2 page 4].

Lemma 1.7.3 (Closed Graph Theorem)

If H is a closed operator on $\mathcal{X}$ and domain $\mathfrak{D}(H)$ is a closed subspace of $\mathcal{X}$, then H is bounded.

PROOF. See Bachman and Narici, [BN66, page 272]. □

Corollary 1.7.4

If H is closed then

$$\rho(H) = \{z \in \mathbb{C} \ : \ zI - H : \mathfrak{D}(H) \to \mathcal{X} \ \text{ is bijective}\}. \tag{1.4}$$

We write $z - H$ for $zI - H$. $R(z, H) := (z-H)^{-1}$ is called the **resolvent operator** of H. The set $\sigma(H) := \mathbb{C} \setminus \rho(H)$ is called **the spectrum** of H.

Lemma 1.7.5

$R(z, H)$ is a norm holomorphic function of z and satisfies the **resolvent equations**

$$R(z,H) - R(w,H) = -(z-w)R(z,H)R(w,H) \tag{1.5}$$

$$R(z,H)R(w,H) = R(w,H)R(z,H) \tag{1.6}$$

$$\frac{d^n}{dz^n} R(z, H) = (-1)^n R(z, H)^{n+1} \tag{1.7}$$

PROOF. see [Dav95, lemma 1.1.2 page 4]. □

$C(\mathbb{R})$ will denote the algebra of all continuous complex valued functions on $\mathbb{R}$.

In Chapter 2, we characterise the operators with real spectra and slowly increasing resolvents when approaching the spectrum. It turns out that important operators that arise from familiar physical problems, are in this class. We also study properties of these operators and obtain some perturbation results.

Chapter 3 looks at the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$. Among other things, we prove that $C_c^\infty(\mathbb{R})$[5] is dense in this algebra. Important functions like $x \mapsto e^x$ are either in the algebra or can be extended to functions in the algebra.

In Chapter 4, tools developed in Chapter 2 and Chapter 3 are used to define the almost analytic functional calculus for operators with real spectra and slowly growing resolvents when approaching the spectrum. Standard theorems like the spectral mapping theorem are proved.

Finally in Chapter 5, we examine applications for our functional calculus. We look at applications in semigroup theory, and some aspects of L^p spaces. We believe there are many more areas not considered here in which this functional calculus could be found valuable.

[5] $C^p(\mathbb{R})$ are $p-$differentiable complex valued functions on $\mathbb{R}$, while $C^p(\mathbb{R})$ are $p-$differentiable functions on $\mathbb{R}$ with compact support

Chapter 2

Growth Condition on $\|(z-H)^{-1}\|$.

Suppose H is a closed densely defined operator on a Banach space $\mathcal{X}$ whose spectrum is contained in $\mathbb{R}$ and there exists $c > 0$ such that

$$\|(z-H)^{-1}\| \leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}} \tag{2.1}$$

for all $z \notin \mathbb{R}$ and some $\alpha,\ \beta \geq 0$. We will say that H is of $(\alpha,\ \beta) - type\ \mathbb{R}$. Here, we define $\langle\rangle$ by $\langle z\rangle^2 := 1 + |z|^2$ and $\Im z$ denotes the **imaginary part of** z (the **real part of** z will be denoted by $\Re z$) . If instead of (2.1) we have

$$\|(z-H)^{-1}\| \leq c\frac{|z|^{\alpha}}{|\Im z|^{\beta}} \tag{2.2}$$

for all $z \notin \mathbb{R}$ and some $\alpha,\ \beta \geq 0$, we will say that H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$.

Remark 2.0.6

1. Condition (2.2) is stronger than (2.1), since $|z| \leq \langle z\rangle\ \forall z \in \mathbb{C}$. Thus $(\alpha,\ \beta)' - type\ \mathbb{R}$ implies $(\alpha,\ \beta) - type\ \mathbb{R}$. However the converse

is not true.

2. Appendix C has details of the properties of $\langle\rangle$.

2.1 Examples of $(\alpha,\ \beta) - type\ \mathbb{R}$ operators

Proposition 2.1.1

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then H is of $(0,\ 1) - type\ \mathbb{R}$.

PROOF. If $S \subset \mathcal{H}$, denote the closure of S by $\overline{S}$ and its orthogonal complement by $S^{\perp}$. Let the adjoint of an operator H be denoted by H^*, its **kernel** be $\ker(H) := \{f \in \mathfrak{D}(H)$ such that $Hf = 0\}$ and suppose $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned}
2i\Im\langle(z-H)f, f\rangle &= \langle(z-H)f, f\rangle - \overline{\langle(z-H)f, f\rangle} \\
&= \langle zf, f\rangle - \langle Hf, f\rangle - \langle f, zf\rangle + \langle f, Hf\rangle \\
&= z\,\|f\|^2 - \bar{z}\,\|f\|^2 \quad (\text{since } H^* = H) \\
&= (z-\bar{z})\,\|f\|^2 \\
&= 2i\Im z\,\|f\|^2 \\
\text{if and only if } \Im\langle f, (z-H)f\rangle &= \Im z\,\|f\|^2 . \\
\text{Which implies } |\Im z|\,\|f\| &\leq \|(z-H)f\| . \qquad (2.3)
\end{aligned}$$

That is, $z-H$ is bounded from below. Thus $z-H$ is injective, so $\ker(z-H) = \{0\}$. Since H is self adjoint, we have

$$\ker((z-H)^*) = \ker(\bar{z}-H) = \{0\}. \qquad (2.4)$$

But because H is closed densely defined,

$$\overline{\mathfrak{R}(z-H)} = \ker\left((z-H)^*\right)^{\perp}.$$

Therefore using (2.4),

$$\overline{\mathfrak{R}(z-H)} \quad = \quad \{0\}^{\perp} \quad = \quad \mathcal{H}.$$

Conclusion:

1. $(z-H)^{-1}$ exists and is bounded,

2. $\mathfrak{R}(z-H)$ is dense in $\mathcal{H}$

thus $z \in \rho(H)$.

The conclusion of the proposition now follows from (2.3). □

Proposition 2.1.2

Let $H_0 = -\frac{d^2}{dx^2}$ on $L^1(\mathbb{R})$ where
$\mathfrak{D}(H_0) = \{f \in L^1(\mathbb{R}) \ : \ f'' \in L^1(\mathbb{R}), f'$ absolutely continuous$\}$. Then

1. *$(z-H_0)^{-1}$ is a convolution operator, for each $z \notin \mathbb{R}$.*

2. *H_0 is of $(0,\ 1)-type$ $\mathbb{R}$.*

PROOF. Let $z \in \rho(H_0)$. Then

$$\widehat{(z-H_0)^{-1}f}(\zeta) = (z-a(\zeta))^{-1}\widehat{f}(\zeta),\ \zeta \in \mathbb{R}$$

where $\widehat{g}$ denotes Fourier transform of g and $a(\zeta)$ is the symbol of H_0. That is

$$\begin{aligned} a(\zeta) &= -(i\zeta)^2 \\ &= \zeta^2. \end{aligned}$$

So that

$$\begin{aligned} \widehat{(z-H_0)^{-1}f}(\zeta) &= \frac{1}{z-\zeta^2}\widehat{f}(\zeta) \\ &= (2\pi)^{\frac{1}{2}}\widehat{g}(\zeta)\widehat{f}(\zeta) \\ &= \widehat{g*f}(\zeta) \end{aligned}$$

where $g*f$ denotes the convolution of g and f and

$$\widehat{g}(\zeta) := \frac{(2\pi)^{-1/2}}{z-\zeta^2}.$$

Since $\widehat{g}$ decays rapidly enough as $|\zeta| \to \infty$, $\widehat{g} \in L^1(\mathbb{R})$.

$$\text{Thus} \quad (z-H_0)^{-1}f(x) = g*f(x), \qquad f \in L^1(\mathbb{R})$$

where $g \in C_0(\mathbb{R})$[1] will be determined explicitly shortly.

This proves the first part of the proposition.

Next, set

$$f_\lambda(x) := e^{-\lambda|x|}, \lambda \in \mathbb{C}\setminus i\mathbb{R}$$

[1] $C_0^p(\mathbb{R})$ will denote $p-$differentiable functions on $\mathbb{R}$ with $f(x) \to 0$ as $|x| \to \infty$

where $i\mathbb{R} := \{ix : x \in \mathbb{R}\}$. Then,

$$\begin{aligned}
\widehat{f_\lambda}(\zeta) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-\lambda|x|}e^{-ix.\zeta}dx \\
&= (2\pi)^{-1/2}\left\{\int_{-\infty}^{0} e^{x(\lambda-i\zeta)}dx + \int_{0}^{\infty} e^{-x(\lambda+i\zeta)}dx\right\} \\
&= (2\pi)^{-1/2}\left\{\frac{1}{\lambda-i\zeta}+\frac{1}{\lambda+i\zeta}\right\} \\
&= (2\pi)^{-1/2}\left\{\frac{2\lambda}{\lambda^2+\zeta^2}\right\}.
\end{aligned}$$

Now given $z \notin \mathbb{R}$,

$$\begin{aligned}
\widehat{g}(\zeta) &= \frac{(2\pi)^{-1/2}}{z-\zeta^2} \\
&= \frac{-(2\pi)^{-1/2}}{2i\sqrt{z}}\frac{2i\sqrt{z}}{((i\sqrt{z})^2+\zeta^2)} \\
&= \frac{i}{2\sqrt{z}}\widehat{f}_{i\sqrt{z}}(\zeta), \quad \text{since } i\sqrt{z} \notin i\mathbb{R}.
\end{aligned}$$

$$\text{Thus} \quad (2\pi)^{1/2}g(-\zeta) = \check{\widehat{g}}(\zeta) = \frac{i(2\pi)^{1/2}}{2\sqrt{z}}f_{i\sqrt{z}}(-\zeta)$$

(where $\check{f}$ denotes the inverse Fourier transform of f)

$$\text{and hence } g(x) = \frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|}.$$

Next

$$\begin{aligned}
\|(z-H_0)^{-1}\| &= \sup\left\{\|g*f\|_1 : f \in L^1, \|f\|_1 = 1\right\} \\
&\leq \|g\|_1 \\
&\quad (\text{since } \|g*f\|_1 \leq \|g\|_1\,\|f\|_1\, f, g \in L^1) \\
&= \int_{-\infty}^{\infty}\left|\frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|}\right| dx.
\end{aligned}$$

By means of a change of variable and reflectional symmetry we conclude that

$$\begin{aligned}
\|(z-H_0)^{-1}\| &= 2\int_0^\infty \left|\frac{i}{2\sqrt{z}}e^{-i\sqrt{z}|x|}\right| dx \\
&= \frac{1}{|\sqrt{z}|}\int_0^\infty e^{-\Re(i\sqrt{z})r}dr \\
&\le \frac{1}{|\sqrt{z}|}\frac{1}{|\Re(i\sqrt{z})|} \\
&= \frac{1}{|\sqrt{z}|\,|\Im\sqrt{z}|} \\
&= \frac{1}{|z|^{1/2}\left|\frac{|z|-\Re z}{2}\right|^{1/2}} \\
&= \frac{\left|\frac{|z|+\Re z}{2}\right|^{1/2}}{|z|^{1/2}\,\frac{\left||z|^2-(\Re z)^2\right|^{1/2}}{2}} \\
&\le \frac{2\,|z|^{1/2}}{|z|^{1/2}\,|z|\left|1-\left|\frac{\Re z}{|z|}\right|^2\right|^{1/2}} \\
&= \frac{2}{|z|\,|1-\cos^2\theta|^{1/2}} \\
&\quad \text{(where } \theta := \arg z) \\
&= \frac{2}{|z|\,|\sin\theta|} \\
&= \frac{2}{|\Im z|}.
\end{aligned}$$

□

Theorem 2.1.3

Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$, and

$$\|e^{iHt}\| \le C(1+|t|)^\alpha, \tag{2.5}$$

where α is an non-negative integer. Then H is of $(\alpha,\ \alpha+1)-type$ $\mathbb{R}$.

PROOF. Let $z \notin \mathbb{R}$. Since $(z-iH)^{-1} = \begin{cases} \int_0^\infty e^{-zt}e^{iHt}dt & \Re z > 0 \\ -\int_{-\infty}^0 e^{-zt}e^{iHt}dt & \Re z < 0 \end{cases}$ (see Bratteli and Robinson [BR87, Proposition 3.16]). Therefore for z with $\Re z > 0$ we have

$$\begin{aligned} \left\|(z-iH)^{-1}f\right\| &\leq \int_0^\infty e^{-|\Re z|t}\left\|e^{iHt}f\right\|dt \\ &\leq C\|f\|\int_0^\infty e^{-|\Re z|t}(1+t)^\alpha dt. \end{aligned}$$

If $\Re z < 0$, put $s=-t$ then

$$\begin{aligned} \left\|-(z-iH)^{-1}f\right\| &\leq \|f\|\int_0^\infty |e^{zs}|\left\|e^{-iHs}\right\|ds \\ &\leq \int_0^\infty e^{-|\Re z|s}\left\|e^{-iHs}f\right\|ds \\ &\leq C\|f\|\int_0^\infty e^{-|\Re z|s}(1+s)^\alpha ds. \end{aligned}$$

But

$$\begin{aligned} \int_0^\infty e^{-|\Re z|t}(1+t)^\alpha dt &= \left[-\frac{1}{|\Re z|}e^{-|\Re z|t}(1+t)^\alpha\right]_0^\infty - \int_0^\infty -\frac{1}{|\Re z|}e^{-|Re z|t}\alpha(1+t)^{\alpha-1}dt \\ &= \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|}\int_0^\infty e^{-|\Re z|t}(1+t)^{\alpha-1}dt \\ &= \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|}\left[\frac{1}{|\Re z|} + \frac{\alpha-1}{|\Re z|}\int_0^\infty e^{-|\Re z|t}(1+t)^{\alpha-1}dt\right] \\ &= \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|^2} + \frac{\alpha(\alpha-1)}{|\Re z|^2}\int_0^\infty e^{-|\Re z|t}(1+t)^{\alpha-2}dt \\ &\vdots \quad \vdots \\ &= \frac{1}{|\Re z|} + \frac{\alpha}{|\Re z|^2} + \frac{\alpha(\alpha-1)}{|\Re z|^3} + \ldots + \frac{\alpha!}{|Re z|^{\alpha+1}}. \end{aligned}$$

Consequently we may conclude that

$$\begin{aligned}
\|(z-iH)^{-1}f\| &\leq C\|f\|\frac{1}{|\Re z|}\left[1+\frac{\alpha}{|\Re z|^1}+\frac{\alpha(\alpha-1)}{|\Re z|^2}+\ldots+\frac{\alpha!}{|\Re z|^\alpha}\right]\\
&= C\|f\|\frac{1}{|\Re z|}\left[1+\frac{\alpha}{|\Re z|}+\frac{\alpha(\alpha-1)}{|\Re z|^2}+\ldots+\frac{\alpha!}{|\Re z|^\alpha}\right]\\
&\leq C\|f\|\frac{\alpha!}{|\Re z|}\left[1+\frac{\alpha}{1!\,|\Re z|}+\frac{\alpha(\alpha-1)}{2!\,|\Re z|^2}+\ldots+\frac{\alpha!}{\alpha!\,|\Re z|^\alpha}\right]\\
&= C\|f\|\frac{\alpha!}{|\Re z|}(1+\frac{1}{|\Re z|})^\alpha\\
&\leq 2^{\alpha/2}C\alpha!\frac{\langle z\rangle^\alpha}{|\Re z|^{\alpha+1}}\|f\|
\end{aligned}$$

where we have used Hölder's inequality to obtain
$1+|\Re z|\leq\sqrt{2}(1+|\Re z|^2)^{1/2}\leq\sqrt{2}\langle z\rangle$.

Now put $w:=i^{-1}z$. Then $w\notin\mathbb{R}$ and

$$\begin{aligned}
\|(w-H)^{-1}f\| &= \|i(z-iH)^{-1}f\|\\
&= \|(z-iH)^{-1}f\|\\
&\leq 2^{\alpha/2}C\alpha!\frac{\langle z\rangle^\alpha}{|\Re z|^{\alpha+1}}\|f\|\\
&= 2^{\alpha/2}C\alpha!\frac{\langle w\rangle^\alpha}{|\Im w|^{\alpha+1}}\|f\|.
\end{aligned}$$

□

REMARK 2.1.4

1. Note that the converse of theorem 2.1.3 is false. A counter example is the following:

   Let $H_0=-\frac{d^2}{dx^2}$ on $L^1(\mathbb{R})$. Then by proposition 2.1.2, $(z-H_0)^{-1}$ is a convolution operator, for each $z\notin\mathbb{R}$ and is of $(0,\ 1)-\mathit{type}\ \mathbb{R}$.

   However, operators e^{iH_0t}, are unbounded for all $t\neq0$, see for example [BTW75, page 27].

2. Since the map $z \rightarrow e^{itz}$ is in $H^\infty(\Omega_\epsilon)$,

   where $\Omega_\epsilon := \{z \in \mathbb{C} : |\Im z| < \epsilon\}$ for some $\epsilon > 0$, one may conjecture that the conclusion of theorem 2.1.3 holds even for unbounded operators whose spectra lie in Ω_ϵ and admit the bound (2.1.3). This problem remains open.

By a **Schrödinger** operator, we mean a partial differential operator on $\mathbb{R}^N$ of the form

$$H := -\frac{1}{2}\Delta + V$$

with $\Delta := \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and V a real valued measurable function on $\mathbb{R}^N$, called the **potential**. Properties of a Schrödinger operator are essentially determined by properties of the potential. For our purpose (which is quite general) we will select potentials as described below.

Definition 2.1.5

A real valued measurable function V on $\mathbb{R}^N$ is said to lie in **Kato class**, K_N if and only if

1. For $N \geq 3$

$$\lim_{a \to 0}\left[\sup_x \int_{|x-y|\leq a} |x-y|^{-(N-2)} |V(y)| \, d^N y\right] = 0.$$

2. For $N = 2$

$$\lim_{a \to 0}\left[\sup_x \int_{|x-y|\leq a} \ln\left\{|x-y|^{-1}\right\} |V(y)| \, d^2 y\right] = 0.$$

3. For $N = 1$

$$\sup_x \int_{|x-y|\leq 1} |V(y)|\, dy < \infty.$$

We say V is in K_N^{loc} if $V\chi_R \in K_N$ for all $R > 0$, where χ_R is the characteristic function of $\{x : |x| \leq R\}$.

A Schrödinger operator, $H := -\Delta + V$ on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$ considered in this study, has potential V such that $V_+ \in K_N^{loc}$ and $V_- \in K_N$. [2] For more details on properties of these operators see B. Simon [Sim82]. The spectrum of the Schrödinger operator with potential chosen as outlined, is real and independent of p (see, Hempel and Voigt [HV86][Prop. 4.3(a)]). The operators $e^{i\Delta t}$ are however unbounded on $L^p(\mathbb{R}^N)$ for all $t \neq 0$, $N \geq 1$ and for all $p \neq 2$ in the range $[1, \infty]$ [BTW75, page 27].

Theorem 2.1.6

Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then $H+\lambda$ is $(N,\ N+1)' -$ type $\mathbb{R}$ for $\lambda > 0$ large enough.

PROOF. See [Pan90]. □

Theorem 2.1.7

Let H be a Schrödinger operator on $L^p(\mathbb{R}^N)$ then H is $(\alpha,\ \alpha+1)' -$ type $\mathbb{R}$ for $\alpha := N\left|\frac{1}{p} - \frac{1}{2}\right|$.

PROOF. See S. Nakamura [Nak94, Theorem 8]. See also note 4 at the end of this chapter. □

[2] $V_-(x) := \max\{0, V(x)\}$ and $V_+(x) := \min\{0, V(x)\}$

Theorem 2.1.8

Let H be an $n \times n$ matrix with real eigenvalues. Then H is of $(n-1,\ n)-$type $\mathbb{R}$. H of $(0,\ 1)-$type $\mathbb{R}$ if and only if it is diagonalizable.

PROOF. Consider an $n \times n$ tridiagonal matrix :

$$A = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a_{n-1} & b_{n-1} \\ 0 & & & c_{n-1} & a_n \end{pmatrix} \tag{2.6}$$

with either $c_i = 0$ for all i or $b_i = 0$ for all i. Let $\mathrm{cof}(A)$ denote the matrix of cofactors of A, that is $\mathrm{cof}(A) = (d_{ij})$ where d_{ij} is the $ij - th$ cofactor of A, given by

$$d_{ij} = \begin{cases} (-1)^{i+j}(a_1 \dots a_{j-1})(b_j \dots b_{i-1})(a_{i+1} \dots a_n) & , \quad i > j \\ (a_1 \dots a_{i-1})(a_{i+1} \dots a_n) & , \quad i = j \\ (-1)^{i+j}(a_1 \dots a_{i-1})(c_i \dots c_{j-1})(a_{j+1} \dots a_n) & , \quad i < j. \end{cases}$$

Inspired by $\prod_{j=m}^{n} x = x^{n-m+1}$, we assume $\prod_{j=m}^{n} s_j = 1$ for any $s \in \mathbb{R}$, whenever $n - m = -1$.

Expanding along the first column we get the determinant of A,

$$\begin{aligned} \det(A) &= (a_1 \dots a_n) - c_1 b_1 (a_3 \dots a_n) \\ &= (a_1 a_2 - c_1 b_1) a_3 \dots a_n \end{aligned}$$

Since either $c_1 = 0$ or $b_1 = 0$,

$$\det(A) = \prod_{i=1}^{n} a_i. \tag{2.7}$$

Next, suppose $\lambda_1, \ldots, \lambda_n$ are the real eigenvalues of H and $z \notin \mathbb{R}$. Then H has the Jordan canonical form

$$H = \begin{pmatrix} \lambda_1 & b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & b_{n-1} \\ 0 & & & \lambda_n \end{pmatrix}$$

where $b_\nu = 0$ or 1 for all ν.

Thus

$$(z-H) = \begin{pmatrix} z-\lambda_1 & -b_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -b_{n-1} \\ 0 & & & z-\lambda_n \end{pmatrix}.$$

This is a special case of (2.6) with $c_1 = c_2 = \ldots = c_{n-1} = 0$. Hence

$$\begin{aligned} \det(z-H) &= \prod_{\nu=1}^{n}(z-\lambda_\nu) \qquad \text{(by (2.7))} \\ &\neq 0 \end{aligned}$$

(since $z \neq \lambda_\nu$ for all ν).

Thus $\mathrm{cof}(z-H) = (d_{ij})$ where

$$d_{ij} = \begin{cases} (-1)^{i+j}(z-\lambda_1)\dots(z-\lambda_{j-1})(-b_j)\dots(-b_{i-1})(z-\lambda_{i+1})\dots(z-\lambda_n) & , \ i>j \\ \prod_{\nu\neq i}^{n}(z-\lambda_\nu) & , \ i=j \\ 0 & , \ i<j. \end{cases}$$

If $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm and A^T is the transpose of the matrix A, then by the above together with the fact that $(z-H)^{-1} = (\det(z-H))^{-1}(\mathrm{cof}(z-H))^T$ we see that

$$\begin{aligned} \left\|(z-H)^{-1}\right\|_{HS}^2 &= \left\|\left[(z-H)^{-1}\right]^T\right\|_{HS}^2 \\ &= \left\|\frac{\mathrm{cof}(z-H)}{\det(z-H)}\right\|_{HS}^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{d_{ij}}{\det(z-H)}\right|^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{\prod_{\substack{1\leq\nu\leq j-1 \\ i+1\leq\nu\leq n}}(z-\lambda_\nu)\cdot\prod_{j\leq\nu\leq i-1}(-b_\nu)}{\prod_{\nu=1}^{n}(z-\lambda_\nu)}\right|^2 \\ &\quad \text{(since } d_{ij}=0 \text{ for all } i<j\text{)}. \end{aligned}$$

Here we have used the fact that $\|A\|_{HS}^2 = \sum_1^n \|Ae_i\|^2$ for any orthonormal system $e_1, \dots, e_n$.

Thus

$$\left\|(z-H)^{-1}\right\|_{HS}^2 = \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\prod_{j\leq\nu\leq i}\frac{1}{(z-\lambda_\nu)}\cdot\prod_{j\leq\nu\leq i-1}(-b_\nu)\right|^2 \quad (2.8)$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{i}\prod_{j\leq\nu\leq i}\left|\frac{1}{z-\lambda_\nu}\right|^2 \quad (2.9)$$

(since $|-b_\nu| \leq 1$ for all ν).

Now, recall the **distance of z from the spectrum** $\sigma(H)$ is

$$\text{dist}\,(z,\ \sigma(H)) := \inf\{|z-w|\ :\ z \in \sigma(H)\}. \tag{2.10}$$

In this case $\sigma(H) = \{\lambda_1, \ldots, \lambda_n\}$, and we can find $\lambda \in \sigma(H)$ such that

$$\text{dist}\,(z,\ \sigma(H)) = |z-\lambda|\,. \tag{2.11}$$

So, $|z-\lambda| \le |z-\lambda_i|$ for all i and

$$\|(z-H)^{-1}\|_{HS}^2 \le \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{1}{z-\lambda}\right|^{2(i-j+1)}. \tag{2.12}$$

Case 1: $\left|\frac{1}{z-\lambda}\right|^2 \le 1$.

$$\begin{aligned}
\|(z-H)^{-1}\|_{HS}^2 &\le \left|\frac{1}{z-\lambda}\right|^2 \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{1}{z-\lambda}\right|^{2(i-j)} \\
&\le \left|\frac{1}{z-\lambda}\right|^2 \sum_{i=1}^{n}\sum_{j=1}^{i} 1 \\
&= \left|\frac{1}{z-\lambda}\right|^2 \sum_{i=1}^{n} i \\
&= \left|\frac{1}{z-\lambda}\right|^2 \frac{n(n+1)}{2}.
\end{aligned}$$

Hence

$$\|(z-H)^{-1}\|_{HS} \le \left|\frac{1}{z-\lambda}\right|\sqrt{\frac{n(n+1)}{2}} \le \frac{\sqrt{\frac{n(n+1)}{2}}}{\Im z} \tag{2.13}$$

Case 2: $\left|\frac{1}{z-\lambda}\right|^2 > 1$.

$$\begin{aligned}
\|(z-H)^{-1}\|_{HS}^2 &\leq \sum_{i=1}^{n}\sum_{j=1}^{i}\left|\frac{1}{z-\lambda}\right|^{2(i-j+1)} \quad \text{(from (2.12))}\\
&\leq \sum_{i=1}^{n}\left|\frac{1}{z-\lambda}\right|^{2i}\cdot i\\
&\leq \left|\frac{1}{z-\lambda}\right|^{2n}\sum_{i=1}^{n} i\\
&= \left|\frac{1}{z-\lambda}\right|^{2n}\frac{n(n+1)}{2}\\
&\leq \frac{n(n+1)}{2}|\Im z|^{-2n}. \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14) we observe that in any case

$$\|(z-H)^{-1}\|_{HS} \leq \sqrt{\frac{n(n+1)}{2}}\cdot\frac{\langle z\rangle^{n-1}}{|\Im z|^n} \quad \text{for all } z \notin \mathbb{R}. \qquad (2.15)$$

Since all norms in a finite dimensional space are equivalent, we conclude that

$$\|(z-H)^{-1}\| \leq c\frac{\langle z\rangle^{n-1}}{|\Im z|^n} \quad \text{for all } z \notin \mathbb{R} \text{ and some } c>0. \qquad (2.16)$$

If H is diagonalizable, then

$$(z-H) = \begin{pmatrix} z-\lambda_1 & & 0\\ & \ddots & \\ 0 & & z-\lambda_n \end{pmatrix}.$$

This is a special case of (2.6) with $c_i = b_i = 0$ for all $i = 2, 1 \ldots n-1$. In this case $\mathrm{cof}(z-H) = (d_{ij})$ where

$$d_{ij} = \begin{cases} 0 & , \quad i > j \\ \prod_{\nu \neq i}^{n} (z - \lambda_\nu) & , \quad i = j \\ 0 & , \quad i < j \end{cases}$$

and

$$\begin{aligned} \left\|(z-H)^{-1}\right\|_{HS}^2 &\leq \sum_{i=1}^{n} \left| \frac{\prod_{\nu \neq i}(z-\lambda_\nu)}{\prod_{\nu=1}^{n}(z-\lambda_\nu)} \right|^2 \\ &= \sum_{i=1}^{n} \left| \frac{1}{z-\lambda_i} \right|^2 \\ &\leq \left| \frac{1}{z-\lambda} \right|^2 n \end{aligned}$$

with λ chosen as in (2.11).

Therefore

$$\begin{aligned} \left\|(z-H)^{-1}\right\|_{HS} &\leq \frac{\sqrt{n}}{|z-\lambda|} \\ &\leq \frac{\sqrt{n}}{|\Im z|}. \end{aligned} \tag{2.17}$$

Thus,

$$\left\|(z-H)^{-1}\right\| \leq \frac{c}{|\Im z|} \text{ for some } c > 0. \tag{2.18}$$

Conversely, if H is not diagonalizable, then from (2.8) we have

$$\left\|(z-H)^{-1}\right\|_{HS}^2 = \sum_{i=1}^{n} \sum_{j=1}^{i} \left| \prod_{j \leq \nu \leq i} \frac{1}{(z-\lambda_\nu)} \cdot \prod_{j \leq \nu \leq i-1} (-b_\nu) \right|^2$$

with $b_k \neq 0$ for some $k \in \{1, \dots n-1\}$. Thus

$$\|(z-H)^{-1}\|_{HS}^2 = \sum_{i=1}^{n} \left|\frac{1}{z-\lambda_i}\right|^2 + \sum_{i=2}^{n}\sum_{j=k_i}^{i-1} \left|\prod_{j\le\nu\le i-1} \frac{-b}{(z-\lambda_\nu)}\right|^2 .$$

Where $k_i := \min\{j \ : \ b_j, b_{j+1}, \dots, b_{i-1} \neq 0\}$. But $|\Im z| = |\Im(z-\lambda_i)|$ for all $i = 1, \dots, n$ and $|-b_j| = 1$ for all $k_i \le j \le i-1$. So if we set $\Re(z+\lambda) := \max\{\Re(z-\lambda_i) \ : \ i = 1, \dots, n\}$ then

$$\begin{aligned}
|\Im z|^2 \|(z-H)^{-1}\|_{HS}^2 &> \sum_{i=1}^{n} \frac{|\Im z|^2}{[|\Im z| + |\Re(z-\lambda)|]^2} \\
&\quad + \sum_{i=2}^{n} \frac{|\Im z|^2}{[|\Im z| + |\Re(z-\lambda)|]^2} \sum_{j=k_i}^{i-1} \frac{1}{[|\Im z| + |\Re(z+\lambda)|]^{a_j}} \\
&\quad \text{with } \ a_j \ge 2 \text{ for all } \ j. \\
&= nK + \sum_{i=2}^{n} K \sum_{j=k_i}^{i-1} \frac{1}{[|\Im z| + |\Re(z+\lambda)|]^{a_j}} \qquad (2.19) \\
\text{where } \ K &:= \frac{|\Im z|^2}{[|\Im z| + |\Re(z-\lambda)|]^2}.
\end{aligned}$$

$K \to 0$ as $\Im z \to 0$ but $\frac{1}{[|\Im z| + |\Re(z+\lambda)|]^{a_j}} \to \infty$ a_i times faster, as $\Im z \to 0$ for any fixed $\Re(z+\lambda)$. Therefore it follows from (2.19) that there is no $D > 0$ such that $|\Im z|^2 \|(z-H)^{-1}\|_{HS}^2 \le D$ for all $z \notin \mathbb{R}$. □

2.2 Properties of $(\alpha,\ \beta)-type\ \mathbb{R}$ operators.

Definition 2.2.1

A C_0- Semigroup T on $\mathcal{X}$ is a family of operators

$$T := \{T(t) \ : t \in \mathbb{R}^+\} \subseteq \mathfrak{B}(\mathcal{X})$$

satisfying

$$T(t)T(s) = T(t+s) \quad \forall\, t, s \in \mathbb{R}^+ \tag{2.20}$$

$$T(0) = I \tag{2.21}$$

$$T(.)f \in C(\mathbb{R}^+, \mathcal{X})\ \forall f \in \mathcal{X}. \tag{2.22}$$

Further, T is said to be a **contraction semigroup** if for each $t \in \mathbb{R}^+$, $T(t)$ is a contraction, i.e. $\|T(t)\| \leq 1$.

Definition 2.2.2

If T is a C_0-Semigroup on $\mathcal{X}$, then the **(infinitesimal) generator**, H of T is defined by the formula

$$Hf := \lim_{t \to 0} \frac{T(t)f - f}{t} = \frac{d}{dt}T(t)f|_{t=0}, \tag{2.23}$$

with maximal domain, that is $\mathfrak{D}(H)$ is the set of all $f \in \mathcal{X}$ for which the limit (2.23) exists.

Lemma 2.2.3 (Hille - Yosida Theorem)

H is a generator of a C_0-contraction semigroup if and only if H is closed, densely defined and for each $\lambda > 0, \lambda \in \rho(H)$ and $\|(\lambda - H)^{-1}\| \leq \lambda^{-1}$.

PROOF. See [Gol85, pages 16 – 17]. □

Lemma 2.2.4

If $\{T(t)\}_{t\geq 0}$ *is a strongly continuous semigroup with generator* H, *then it can be extended to a strongly continuous one-parameter group* $\{U(t)\}_{t\in\mathbb{R}}$ *if and only if* $-H$ *generates a strongly continuous semigroup* $\{S(t)\}_{t\geq 0}$. *In that case* $\{U(t)\}_{t\in\mathbb{R}}$ *is obtained as*

$$U(t) := \begin{cases} T(t) & for\ t \geq 0 \\ S(-t) & for\ t \leq 0. \end{cases}$$

PROOF. See Davies [Dav80, Prop. 1.14]. □

Theorem 2.2.5

H *is of* $(0,\ 1) - type\ \mathbb{R}$ *with the constant* $c = 1$ *if and only if* iH *is a generator of a one-parameter group of isometries on* $\mathcal{X}$.

PROOF. First,

necessity: Assume H is of $(0,\ 1) - type\ \mathbb{R}$.

Clearly $\pm iH$ are closed densely defined (by the hypothesis on H). Suppose $\lambda > 0$. Then $\lambda \in \rho(\pm iH)$ since $\sigma(H) \subset \mathbb{R}$, and

$$\begin{aligned} \|(\lambda \pm iH)^{-1}\| &= \left\|\frac{1}{i}(\frac{\lambda}{i} \pm H)^{-1}\right\| \\ &= \|(-i\lambda \pm H)^{-1}\| \\ &\leq |\Im(\pm i\lambda)|^{-1} \\ &= \lambda^{-1}. \end{aligned}$$

Thus by Hille - Yosida theorem (Lemma 2.2.3), $\pm iH$ are generators of contraction semigroups. Finally, the conclusion follows by invoking Lemma 2.2.4.

sufficiency: Suppose iH is a generator of a group of isometries $\{T(t)\}$.

Then for all $w \in \mathbb{C}$ with $\Re w \neq 0$, $w \in \rho(iH)$ and

$$(\lambda - iH)^{-1} = \begin{cases} \int_0^\infty T(t)e^{-\lambda t}dt & \text{if } \Re\lambda > 0 \\ -\int_\infty^0 T(t)e^{-\lambda t}dt & \text{if } \Re\lambda < 0 \end{cases}$$

(see Bratteli and Robinson, [BR87, Proposition 3.16]).

From this

$$\begin{aligned} \left\|(\lambda - iH)^{-1}\right\| &\leq \int_0^\infty e^{-|\Re\lambda| t}\, \|T(t)\|\, dt \\ &\leq \int_0^\infty e^{-|\Re\lambda| t} dt \\ &= |\Re(\lambda)|^{-1}. \end{aligned}$$

Now put $z := \frac{\lambda}{i}$.

□

Theorem 2.2.6

If H is of $(\alpha,\ \alpha+1)'$ − type $\mathbb{R}$ then λH is also of $(\alpha,\ \alpha+1)'$ − type $\mathbb{R}$ with the same constant c for all $\lambda > 0$.

Proof. Let $z \notin \mathbb{R}$, then

$$\begin{aligned} \left\|(z - \lambda H)^{-1}\right\| &= \left\|\lambda^{-1}(\frac{z}{\lambda} - H)^{-1}\right\| \\ &= \left|\lambda^{-1}\right| \left\|(\frac{z}{\lambda} - H)^{-1}\right\| \\ &\leq \left|\lambda^{-1}\right| c \left|\Im\frac{z}{\lambda}\right|^{-1} \left(\frac{|\frac{z}{\lambda}|}{|\Im\frac{z}{\lambda}|}\right)^{\alpha} \quad \text{(hypothesis)} \\ &= \left|\lambda^{-1}\right| c\, |\Im z|^{-1}\, |\lambda| \left(\frac{|z|}{|\Im z|}\right)^{\alpha} \\ &= c\, |\Im z|^{-1} \left(\frac{|z|}{|\Im z|}\right)^{\alpha}. \end{aligned}$$

□

REMARK 2.2.7

The type of stability shown in theorem 2.2.6 will be called **scale invariance**.

Theorem 2.2.8

If H is of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ then λH is also of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ with the same constant c for all $0<\lambda<1$.

PROOF. Let $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned}
\left\|(z-\lambda H)^{-1}\right\| &= \left\|\lambda^{-1}\left(\frac{z}{\lambda}-H\right)^{-1}\right\| \\
&= \left|\lambda^{-1}\right| \left\|\left(\frac{z}{\lambda}-H\right)^{-1}\right\| \\
&\leq \left|\lambda^{-1}\right| c \left|\Im \frac{z}{\lambda}\right|^{-1} \left(\frac{\langle\frac{z}{\lambda}\rangle}{\left|\Im \frac{z}{\lambda}\right|}\right)^{\alpha} \quad \text{(hypothesis).}
\end{aligned}$$

$$\begin{aligned}
\text{Thus } \left\|(z-\lambda H)^{-1}\right\| &\leq \left|\lambda^{-1}\right| c \left|\Im z\right|^{-1} |\lambda| \left(\frac{\sqrt{\lambda^2+|z|^2}}{|\Im z|}\right)^{\alpha} \\
&\leq c\left|\Im z\right|^{-1} \left(\frac{\sqrt{1+|z|^2}}{|\Im z|}\right)^{\alpha} \quad \text{(since } \lambda<1) \\
&= c\left|\Im z\right|^{-1} \left(\frac{\langle z\rangle}{|\Im z|}\right)^{\alpha}.
\end{aligned}$$

□

REMARK 2.2.9

The type of stability shown in theorem 2.2.8 will be called **scale sub-invariance**.

Theorem 2.2.10

If H is of $(\alpha,\ \beta)-type\ \mathbb{R}$ then $H+\lambda$ is also of $(\alpha,\ \beta)-type\ \mathbb{R}$ for all $\lambda \in \mathbb{R}$.

PROOF. If $\lambda \in \mathbb{R}$ and $z \notin \mathbb{R}$ then $\Im(z-\lambda) = \Im z$. Therefore, using Lemma C.0.9 we get

$$\begin{aligned}
\left\|(z-(H+\lambda))^{-1}\right\| &= \left\|[(z-\lambda)-H]^{-1}\right\| \\
&\leq c\frac{\langle z-\lambda\rangle^{\alpha}}{|\Im(z-\lambda)|^{\beta}} \quad \text{(hypothesis)} \\
&= c\frac{\langle z-\lambda\rangle^{\alpha}}{|\Im z|^{\beta}} \\
&\leq c_1\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}}
\end{aligned}$$

where $c_1 = c2^{\alpha/2}\langle\lambda\rangle^{\alpha}$. □

If H is of $(\alpha,\ \beta)' - type\ \mathbb{R}$ then it is also of $(\alpha,\ \beta) - type\ \mathbb{R}$ with the converse not true in general, Remark 2.0.6. However the following result provides some sort of converse to this.

Theorem 2.2.11

If λH is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ for all $\lambda > 0$, then H is of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$ with the same constant c.

PROOF. Let $z \notin \mathbb{R}$, then

$$\begin{aligned}
\left\|(z-H)^{-1}\right\| &= \left\|\lambda\,(\lambda z-\lambda H)^{-1}\right\| \\
&= |\lambda|\,\left\|(\lambda z-\lambda H)^{-1}\right\| \\
&\leq |\lambda|\,c\,|\Im(\lambda z)|^{-1}\left(\frac{\langle\lambda z\rangle}{|\Im(\lambda z)|}\right)^{\alpha} \quad \text{(hypothesis)} \\
&= c\,|\Im z|^{-1}\left(\frac{\sqrt{\lambda^{-2}+|z|^{2}}}{|\Im z|}\right)^{\alpha} \quad \text{for all } \lambda > 0.
\end{aligned}$$

Letting $\lambda \to \infty$, we observe that

$$\left\|(z-H)^{-1}\right\| \leq c\,|\Im z|^{-1}\left(\frac{|z|}{|\Im z|}\right)^{\alpha}.$$

□

Conjecture 2.2.12

If H is of $(\alpha,\ \beta)-type\ \mathbb{R}$, $\beta \geq \alpha$ and $0 \notin \sigma(H)$ then H is of $(\alpha+1,\ \beta)'-type\ \mathbb{R}$.

Theorem 2.2.13

Let H be of $(\alpha,\ \beta)-type\ \mathbb{R}$. Then H^2 is of $(\frac{\alpha+\beta-1}{2},\ \beta)-type\ \mathbb{R}$.

PROOF. Given $z \notin \mathbb{R}$, let $\theta_z := \arg z$ (the argument of z). Therefore using (C.1) of Lemma C.0.7, we have

$$\begin{aligned}
\frac{1}{|\Im\sqrt{z}|^2} &= \frac{1}{\frac{1}{2}\,||z| - \Re z|} \\
&= \frac{2}{||z| - |z|\cos\theta_z|} \\
&= \frac{2}{|z|\,|1-\cos\theta_z|} \\
&= \frac{2\,|1+\cos\theta_z|}{|z|\,|\sin^2\theta_z|} \\
&= \frac{2\,|1+\cos\theta_z|}{|\Im z|\,|\sin\theta_z|} \\
&\leq \frac{4}{|\Im z|}\,\frac{1}{|\sin\theta_z|} \\
&= \frac{4\,|z|}{|\Im z|^2}.
\end{aligned}$$

Also $\langle\sqrt{z}\rangle^2 \leq \sqrt{2}\,\langle z\rangle$, Lemma C.0.8.

So, from

$$
\begin{aligned}
2\sqrt{z}(z-H^2)^{-1} &= (\sqrt{z}-H)^{-1} - (-\sqrt{z}-H)^{-1} \\
\text{we get } \left\|(z-H^2)^{-1}\right\| &\leq \frac{1}{2}\left|\frac{1}{\sqrt{z}}\right| \left\{\left\|(\sqrt{z}-H)^{-1}\right\| + \left\|(-\sqrt{z}-H)^{-1}\right\|\right\} \\
&\leq c\left|\frac{1}{\sqrt{z}}\right| \frac{\langle\sqrt{z}\rangle^{\alpha}}{|\Im\sqrt{z}|^{\beta}} \quad \text{(hypothesis)} \\
&\leq \frac{c(\sqrt{2}\langle z\rangle)^{\alpha/2}}{|\sqrt{z}|}\left(\frac{4|z|}{|\Im z|^2}\right)^{\beta/2} \\
&= \frac{c2^{\alpha/4}2^{\beta}\langle z\rangle^{\alpha/2}|z|^{\beta/2-1/2}}{|\Im z|^{\beta}} \\
&\leq \frac{c2^{\alpha/4+\beta}\langle z\rangle^{(\alpha+\beta-1)/2}}{|\Im z|^{\beta}}.
\end{aligned}
$$

□

Proposition 2.2.14

Let A *be of* $(\alpha,\ \alpha+1)-$ *type* $\mathbb{R}$ *with*

$$\left\|(z-A)^{-1}\right\| \leq c_1\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} \qquad \text{for some } c_1>0 \text{ and } \alpha\geq 0$$

and B *of* $(\beta,\ \beta+1)-$ *type* $\mathbb{R}$ *with*

$$\left\|(z-B)^{-1}\right\| \leq c_2\frac{\langle z\rangle^{\beta}}{|\Im z|^{\beta+1}} \qquad \text{for some } c_2>0 \text{ and } \beta\geq 0.$$

Then

$$\left\|(z-A)^{-1}-(z-B)^{-1}\right\| \leq (1+\sqrt{2}c_1)(1+\sqrt{2}c_2)\left\|(i+A)^{-1}-(i+B)^{-1}\right\|\frac{\langle z\rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}.$$

Proof. $(z-A)$ and $(z-A)^{-1}$ commute on $\mathfrak{D}(A)$ and hence $(i+A)$ and $(z-A)^{-1}$ also commute on $\mathfrak{D}(A)$ since by linearity of $(z-A)^{-1}$, for all $g\in\mathfrak{D}(A)$, we have

$$\begin{aligned}(z-A)^{-1}(i+A)g &= (z-A)^{-1}\{(z+i)-(z-A)\}\,g \\ &= (z-i)(z-A)^{-1}g-(z-A)^{-1}(z-A)g \\ &= (z-i)(z-A)^{-1}g-(z-A)(z-A)^{-1}g \\ &= (i+A)(z-A)^{-1}g.\end{aligned}$$

Also, $(i+A)^{-1}$ maps into $\mathfrak{D}(A)$ and hence $(i+A)$ and $(z-A)^{-1}$ commute on $\mathfrak{R}((i+A)^{-1})$.

The operators $D := (i+A)(z-A)^{-1}$, $H := [(i+A)^{-1}-(i+B)^{-1}]$ and $E := (i+B)(z-B)^{-1}$ are defined everywhere and bounded on $\mathcal{X}$ as can be seen by writing $D = (i+z)(z-A)^{-1}-I$ and $E = (i+z)(z-B)^{-1}-I$.

Now for $f \in \mathcal{X}$ we have

$$\begin{aligned}&D\left[(i+A)^{-1}-(i+B)^{-1}\right]Ef \\ &= (i+A)(z-A)^{-1}\left[(i+A)^{-1}-(i+B)^{-1}\right](i+B)(z-B)^{-1}f \\ &= (i+A)(z-A)^{-1}(i+A)^{-1}(i+B)(z-B)^{-1}f-(i+A)(z-A)^{-1}(i+B)^{-1}(i+B)(z-B)^{-1}f \\ &= (z-A)^{-1}(i+A)(i+A)^{-1}(i+B)(z-B)^{-1}f-(i+A)(z-A)^{-1}(z-B)^{-1}f \\ &= (z-A)^{-1}(i+B)(z-B)^{-1}f-(i+A)(z-A)^{-1}(z-B)^{-1}f \\ &= -(z-A)^{-1}[I-(i+z)(z-B)^{-1}]f+[I-(i+z)(z-A)^{-1}](z-B)^{-1}f \\ &= -(z-A)^{-1}f+(i+z)(z-A)^{-1}(z-B)^{-1}f+(z-B)^{-1}f-(i+z)(z-A)^{-1}(z-B)^{-1}f \\ &= -\left[(z-A)^{-1}-(z-B)^{-1}\right]f.\end{aligned}$$

Using lemma C.0.8 we have,

$$\begin{aligned}
\|D\| &= \left\|(i+z)(z-A)^{-1} - I\right\| \\
&\leq 1 + (1+|z|)\left\|(z-A)^{-1}\right\| \\
&\leq 1 + (1+|z|)c_1 \frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} \\
&\leq 1 + \sqrt{2}\,\langle z\rangle\, c_1 \frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} \\
&\leq (1+\sqrt{2}c_1)\frac{\langle z\rangle^{\alpha+1}}{|\Im z|^{\alpha+1}}
\end{aligned}$$

Similarly, $\|E\| \leq (1+\sqrt{2}c_2)\frac{\langle z\rangle^{\beta+1}}{|\Im z|^{\beta+1}}$.

Therefore

$$\left\|(z-A)^{-1} - (z-B)^{-1}\right\| \leq (1+\sqrt{2}c_1)(1+\sqrt{2}c_2)\,\|H\|\,\frac{\langle z\rangle^{\alpha+\beta+2}}{|\Im z|^{\alpha+\beta+2}}.$$

□

Definition 2.2.15

Let $\mathfrak{F}$ be a topological algebra of complex-valued functions, H be an operator on a Banach space $\mathcal{X}$ and there exists complex number $\lambda \in \rho(H)$, such that **the resolvent function** $r_\lambda(z) := (\lambda - z)^{-1} \in \mathfrak{F}$. Then H **has a $\mathfrak{F}$ functional Calculus** (*or admits $\mathfrak{F}$ functional Calculus*) if there exists a continuous homomorphism $f \mapsto f(H)$ from $\mathfrak{F}$ to $\mathfrak{B}(\mathcal{X})$ such that whenever $\lambda \in \rho(H)$ and $r_\lambda \in \mathfrak{F}$, we have $r_\lambda(H) = (\lambda - H)^{-1}$.

Theorem 2.2.16

If H is of $(\alpha,\ \alpha+1)'$ – type $\mathbb{R}$, for some $\alpha > 0$, then H admits $C_0^\infty(\mathbb{R})$ functional calculus.

PROOF. See Balabane et al. [MJ93, Theorem 4.11] □

Notes and remarks on Chapter 2

1. According to H. Tanabe [Tan79], a closed operator T on $\mathcal{X}$ is said to be of *type* ω where $0 \leq \omega < \pi$, if $\operatorname{supp}(T) \subseteq \overline{S_\omega}$ and for $0 < \epsilon < \pi - \omega$, ($S_\omega := \{z : |\arg(z)| < \omega\}$) there exists a positive constant c_ϵ such that $\|(z-T)^{-1}\| \leq \frac{c_\epsilon}{|z|}$, $z \notin \overline{S_{\omega+\epsilon}}$. So operators of *type 0* are a bit restrictive but correspond to $(0,\ 1)-$*type* $\mathbb{R}$ operators in our nomenclature. In both cases, the names indicate the location of the spectrum and the growth condition of the resolvent. DeLaubenfels [DeL93] also uses this idea in the study of $\alpha-$*type* V operators, whose resolvents are polynomially ("degree α") bounded with spectra in V.

2. Proposition 2.1.2(1) appears in Hille and Phillips [HP81] where it is shown that $\sigma(H_0) \subseteq [0,\infty)$ and $-H_0$ is the infinitesimal generator of Gauss-Weierstrass semigroup

$$(G_t f)x = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x-y)e^{-\frac{y^2}{4t}}dy,\ f \in L^p(\mathbb{R})$$

3. In Kato [Kat80, IX, sec. 18] it is shown that

$$\|(z-H_0)^{-1}\| \leq \frac{1}{|z|}\left|\sin^2\frac{1}{2}\arg(z)\right|, \quad z \in \rho(H_0) = \mathbb{C}\setminus[0,\infty),$$

This is our Proposition 2.1.2(2).

4. Theorem 8 in S. Nakamura [Nak94] is stated for $(\alpha,\ \alpha+1)-$*type* $\mathbb{R}$ operators. However, the proof given shows that Schrödinger opera-

tors are of $(\alpha,\ \alpha + 1)' - type\ \mathbb{R}$ which is stronger than the stated result.

5. The following are known results used in this chapter:

Lemma 2.2.3	Lemma 2.2.4	Theorem 2.1.6
Theorem 2.1.7	Theorem 2.2.16	

6. Our contributions in this chapter include:

Proposition 2.1.1	Proposition 2.1.2 (new proof)	
Proposition 2.2.14	Theorem 2.1.3	Theorem 2.1.8
Theorem 2.2.5	Theorem 2.2.6	Theorem 2.2.8
Theorem 2.2.10	Theorem 2.2.11	Theorem 2.2.13

Chapter 3

Smooth functions of rapid descent.

3.1 Preliminaries

We may view self-adjoint operators in Hilbert space as the best understood properly infinite dimensional abstract operators. If we desire to recuperate some of their nice properties without the stringent self-adjointness hypothesis, we are led to a *non-self-adjoint theory* such as Dunford's theory of spectral operators [DS71] with the key tool being the resolution of the identity, or Foias' *theory of Generalised spectral Operators* [Foi60, CF68] with the key tool being the distribution theory. Our basic concept, as in Foias theory, will be functional calculus.

We have already seen that if H is of $(\alpha,\ \alpha+1)' - type\ \mathbb{R}$, for some $\alpha > 0$, then H admits $C_0^\infty(\mathbb{R})$ functional calculus (Theorem 2.2.16). In fact a bounded operator with the spectrum lying in a compact set $V \subset \mathbb{R}$, has $C^\infty(V)$ functional calculus. On the other hand, an operator H acting on a Hilbert space $\mathcal{H}$, admits a $C(\mathbb{R})$ functional calculus if H is self-adjoint. So we are really interested in a large enough intermediate topological algebra $\mathfrak{A}$, with $C_0^\infty(\mathbb{R}) \subset \mathfrak{A} \subseteq C(\mathbb{R})$ such that $(\alpha,\ \beta) - type\ \mathbb{R}$ operators in particular (and spectral operators in general) admit a $\mathfrak{A}$ functional calculus or albeit for a restricted range of α and β.

In this chapter, we construct such an algebra and prove some related results of independent interest. We characterise this algebra fully to enable us construct a functional calculus for $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ operators in Chapter 4.

3.2 The topological algebra $\mathfrak{A}$.

For $\beta \in \mathbb{R}$, we define $\mathfrak{S}^\beta$ to be the space of smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $r \geq 0$ there exists $c_r > 0$ so that

$$|f^{(r)}(x)| := \left|\frac{d^r}{dx^r} f(x)\right| \leq c_r \langle x\rangle^{\beta - r}, \quad \text{for all } x \in \mathbb{R}. \tag{3.1}$$

REMARK 3.2.1

- Observe that $\mathfrak{S}^\beta \mathfrak{S}^\gamma \subseteq \mathfrak{S}^{\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{R}$.
- If $f \in \mathfrak{S}^\beta$ then so is $\bar{f}$ where $\bar{f}(z) := \overline{f(z)}$ for all $z \in \mathbb{C}$.

Define the **translation operator** τ_ϵ on the space of functions $f : \mathbb{R} \to \mathbb{C}$ by $\tau_\epsilon f(x) := f(x + \epsilon)$ for all $x \in \mathbb{R}$ and some $\epsilon \in \mathbb{R}$. Then we have the following Lemma.

Lemma 3.2.2

For $\beta < 0$, the space $\mathfrak{S}^\beta$ is invariant under translation τ_ϵ for $\epsilon > 0$.

PROOF. Let $f \in \mathfrak{S}^\beta$ then by (3.1) we can find $c_r > 0$ such that $\left|\frac{d^r}{dx^r} f(x)\right| \leq c_r \langle x\rangle^{\beta-r}$, for all $x \in \mathbb{R}$.

$$\text{But} \quad \left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| = \left|\frac{d^r}{dx^r} f(x+\epsilon)\right| \leq c_r \langle x+\epsilon\rangle^{\beta-r} \quad \text{by use of the chain rule.}$$

$$\text{Therefore} \quad \langle x\rangle^{r-\beta}\left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| \leq c_r \left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$$

with $\left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$ bounded on $\mathbb{R}$ and the bound goes to 1 as $\epsilon \to 0$, see figure 3.1.

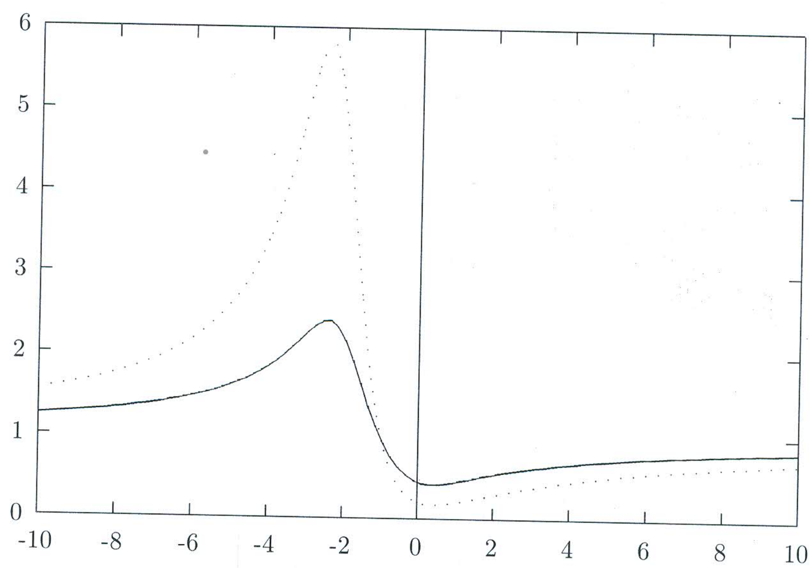

Figure 3.1: Graphs of $\left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$ for various ϵ's

Now set $D_{r,\epsilon} := c_r \sup_{x\in\mathbb{R}} \left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$, then we have

$$\left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| \leq D_{r,\epsilon}\langle x\rangle^{\beta-r}, x \in \mathbb{R}.$$

Thus $\tau_\epsilon f \in \mathfrak{S}^\beta$. □

Theorem 3.2.3

The space

$$\mathfrak{A} := \cup_{\beta<0}\mathfrak{S}^\beta \tag{3.2}$$

is an algebra under pointwise multiplication.

PROOF. Let $f, g \in \mathfrak{A}$ and $\alpha, \lambda \in \mathbb{C}$. Then $f, g \in C^\infty(\mathbb{R})$ and we can find $c_{f,n}, c_{g,n} \in (0,\infty)$ such that

$$\left|\frac{d^n}{dx^n}f(x)\right| \leq \frac{c_{f,n}}{\langle x\rangle^{n-\beta_1}} \quad \text{and} \quad \left|\frac{d^n}{dx^n}g(x)\right| \leq \frac{c_{g,n}}{\langle x\rangle^{n-\beta_2}},$$

for some $\beta_1, \beta_2 < 0$ and all $n \geq 0$. So we have

$$\frac{d^n}{dx^n}(\alpha f(x) + \lambda g(x)) = \alpha\frac{d^n}{dx^n}f(x) + \lambda\frac{d^n}{dx^n}g(x)$$

(by linearity of $\frac{d^n}{dx^n}$). Therefore

$$\begin{aligned}
\left|\frac{d^n}{dx^n}(\alpha f(x) + \lambda g(x))\right| &= \left|\alpha\frac{d^n}{dx^n}f(x) + \lambda\frac{d^n}{dx^n}g(x)\right| \\
&\leq |\alpha|\frac{c_{f,n}}{\langle x\rangle^{n-\beta_1}} + |\lambda|\frac{c_{g,n}}{\langle x\rangle^{n-\beta_2}} \\
&\leq \frac{|\alpha|\, c_{f,n} + |\lambda|\, c_{g,n}}{\langle x\rangle^{n-\beta}} \\
&\quad \text{(where } \beta := \max\{\beta_1, \beta_2\}) \\
&= \frac{c_{f+g,n}}{\langle x\rangle^{n-\beta}}, \quad c_{f+g,n} > 0,\ \beta < 0 \quad \text{for all } n \geq 0.
\end{aligned}$$

Therefore $\alpha f + \lambda g \in \mathfrak{A}$, showing that $\mathfrak{A}$ is linear.

Next, by the Leibniz rule,

$$
\begin{aligned}
\left|\frac{d^n}{dx^n}(f(x)g(x))\right| &= \left|\sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\frac{d^i}{dx^i}f(x)\frac{d^{n-i}}{dx^{n-i}}g(x)\right| \\
&\leq \sum_{i=0}^{n} C_i \langle x\rangle^{\beta_1 - i} \langle x\rangle^{\beta_2-(n-i)} \\
&\quad [\text{ where } C_i := \frac{n!}{i!(n-i)!}\max(c_{f,i}, c_{g,n-i}) \,] \\
&= \langle x\rangle^{\beta_1+\beta_2-n} \sum_{i=0}^{n} C_i \\
&= d_n \langle x\rangle^{\beta_1+\beta_2-n}, \quad d_n > 0.
\end{aligned}
$$

$$\text{Thus } fg \in \mathfrak{A}. \tag{3.3}$$

□

Definition 3.2.4

The **support of** f is the set

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}+ \ : \ f(x) \neq 0\}}.$$

This notion of support of a function will feature prominently in the rest of our work.

The algebra $\mathfrak{A}$ contains the sub-algebra $C_c^\infty(\mathbb{R})$ of all smooth functions with compact support. The completions $\mathfrak{A}_n$ of $\mathfrak{A}$ or $C_c^\infty(\mathbb{R})$ with respect to the norms

$$\|f\|_n := \sum_{r=0}^{n} \int_{-\infty}^{\infty} \left|f^{(r)}(x)\right| \langle x\rangle^{r-1}\, dx \tag{3.4}$$

are also algebras under pointwise multiplication, and much of what we prove below could be extended to these spaces. In fact we have the following.

Lemma 3.2.5

The space $C_c^\infty(\mathbb{R})$ is dense in $\mathfrak{A}$ for each norms $\|\cdot\|_{n+1}$.

PROOF. Suppose that $f \in \mathfrak{S}^\beta$ for some $\beta < 0$. Let $\phi \in C_c^\infty$ such that

$$\phi(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2, \end{cases}$$

see Theorem B.0.6 in the Appendix.

Set $\phi_m(s) := \phi(s/m)$ and $f_m := \phi_m f$. If $n \geq 1$ then

$$\begin{aligned} \|f - f_m\|_{n+1} &= \sum_{r=0}^{n+1} \int_{-\infty}^{\infty} \left| \frac{d^r}{dx^r}\{f(x)(1-\phi_m(x))\} \right| \langle x \rangle^{r-1}\, dx. \\ &\leq \sum_{r=0}^{n+1} \int_{-\infty}^{\infty} \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \left| \frac{d^k}{dx^k} f(x) \right| \left| \frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x)) \right| \langle x \rangle^{r-1}\, dx, \end{aligned}$$

by the Leibniz formula.

We make the following observations:

1. For k=r,

$$\begin{aligned} \left| \frac{d^k}{dx^k} f(x) \right| \left| \frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x)) \right| \langle x \rangle^{r-1} &= \left| \frac{d^r}{dx^r} f(x) \right| |1-\phi_m(x)| \langle x \rangle^{r-1} \\ &\leq c \langle x \rangle^{\beta - r} |1-\phi_m(x)| \langle x \rangle^{r-1} \\ &= c\,|1-\phi_m(x)| \langle x \rangle^{\beta-1} \end{aligned}$$

for some $c \in (0, \infty)$.

2. supp $\left(\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right) \subset \{x : m \leq |x| \leq 2m\}$ for $k < r$, while supp $(1-\phi_m(x)) \subset \{x : |x| > m\}$.

3. For $s \geq 1$ we have the bound,

$$\left|\frac{d^s}{dx^s}(1-\phi_m(x))\right| \leq c_s m^{-s}\chi_m(x) \leq c'_s \langle x\rangle^{-s} \chi_m(x)$$

valid for $m \geq 2$, where χ_m is the characteristic function of $\{x : m \leq |x| \leq 2m\}$.

4. From 3, we conclude that $\left|\frac{d^k}{dx^k}f(x)\right| \left|\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right| \langle x\rangle^{r-1} \leq cc'_s \langle x\rangle^{\beta-1} \chi_m$ for $0 \leq k < r$.

These yield

$$\|f - f_m\|_{n+1} \leq \tilde{c}\sum_{r=0}^{n+1}\int_{|x|>m} \langle x\rangle^{\beta-1}\, dx \quad \text{for some } \tilde{c} > 0$$

which converges to 0 as $m \to \infty$. □

It is important for application and our proofs in Chapter 4 that the functions in $\mathfrak{A}$ need not be $\mathbb{R}$-integrable.

3.3 Functions that lie in $\mathfrak{A}$.

Definition 3.3.1

Let $\mathbf{B}_b(\mathbb{R})$ denote the space of bounded complex valued functions on $\mathbb{R}$ with the uniform norm. A set $\mathfrak{F} \subset \mathbf{B}_b(\mathbb{R})$ is said to **distinguish between points of** $\mathbb{R}$ if for each pair $s, t \in \mathbb{R}$ with $s \neq t$, there is a function $f \in \mathfrak{F}$ such that $f(s) \neq f(t)$.

Lemma 3.3.2 (Stone-Weierstrass Theorem)

Let $\mathfrak{F}$ be a closed sub-algebra of $C_0(\mathbb{R})$, with the supremum norm $\|\cdot\|_\infty$, and closed with respect to complex conjugation. Then $\mathfrak{F} = C_0(\mathbb{R})$ if and only if $\mathfrak{F}$ distinguishes between points of $\mathbb{R}$ and for each finite point of $\mathbb{R}$, contains a function which does not vanish there.

Proof. See for example Dunford and Schwartz [DS58, page 274]. □

Example 3.3.3

Let $w \in \mathbb{C} \setminus \mathbb{R}$ and set $r_w := \frac{1}{w-x}$, $x \in \mathbb{R}$ then $r_w \in \mathfrak{A}$.

Indeed

$$\frac{d^n}{dx^n} r_w(x) = \frac{n!}{(w-x)^{n+1}} \quad \text{for all} \quad n \geq 0$$

showing that r_w is smooth on $\mathbb{R}$.

Next, set $\Gamma := \mathbb{R}$ and $z := x$, in the notations of lemma C.0.10, (see Appendix C). Then using (C.8),

$$\begin{aligned} \left|\frac{d^n}{dx^n} r_w(x)\right| &= \frac{n!}{|w-x|^{n+1}} \\ &\leq \frac{2^{(n+1)/2} n!\, \langle w\rangle^{n+1}}{\left(\sqrt{\beta_0}\, \langle x\rangle\right)^{n+1}} \\ &= \frac{n!\left(\sqrt{2}\,\langle w\rangle\right)^{n+1}}{(\beta_0)^{(n+1)/2}} \langle x\rangle^{-1-n} \quad \text{for all } x \in \mathbb{R}, \text{ and all } n \geq 0. \end{aligned} \tag{3.5}$$

With $\beta_0 \in (0, 1 - \langle \Im w\rangle^{-1})$ in this case.

Thus $r_w \in \mathfrak{S}^{-1} \subset \mathfrak{A}$. □

Corollary 3.3.4

$\mathfrak{A}$ is dense in $C_0(\mathbb{R})$ with respect to uniform norm.

PROOF. Note that $\mathfrak{A}$ is closed with respect to complex conjugation, see Remark 3.2.1.

For $x, y \in \mathbb{R}$,

$$x \neq y \iff r_w(x) \neq r_w(y) \quad \text{for some } w \notin \mathbb{R}.$$

But from example 3.3.3, $r_w \in \mathfrak{A}$ for all $w \notin \mathbb{R}$. Thus $\mathfrak{A}$ distinguishes points of $\mathbb{R}$. Therefore by Stone-Weierstrass Theorem (lemma 3.3.2), $\overline{\mathfrak{A}} = C_0(\mathbb{R})$ with respect to the uniform norm. □

We are now in a position to prove the following perturbation result:

Lemma 3.3.5

If $f \in \mathfrak{A}$ and $c, w \in \mathbb{C}$ with $\Im w \neq 0$ then $(x+c)(w-x)^{-1}f,\ (f+c)(w-x)^{-1} \in \mathfrak{A}$.

PROOF.

$$(x+c)(w-x)^{-1}f = \{-1 + (c+w)(w-x)^{-1}\}f = -f + (c+w)r_w f.$$

(where $r_w := (w-x)^{-1}$), and

$$(f+c)(w-x)^{-1} = fr_w + cr_w.$$

Hence the result follows from Example 3.3.3 and Theorem 3.2.3. □

Theorem 3.3.6

For an arbitrary $t \in \mathbb{R}$ and $f \in \mathfrak{A}$, define $\acute{f}_t$ by

$$\acute{f}_t(x) := \begin{cases} \frac{f(t)-f(x)}{t-x}, & x \neq t \\ f'(t), & x = t. \end{cases}$$

Then $\acute{f}_t \in \mathfrak{A}$.

PROOF. For $x \neq t$,

$$\begin{aligned}\acute{f}_t^{(m)}(x) &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} f^{(k)}(x)(-1)^{m-k}(m-k)!(t-x)^{k-m-1} \\ &+(m!)f(t)(t-x)^{-m-1}(-1)^m.\end{aligned}$$

$$\begin{aligned}\text{Thus } \left|\acute{f}_t^{(m)}(x)\right| &\leq \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left|f^{(k)}(x)\right| |t-x|^{k-m-1} + m! \left|f(t)\right| |t-x|^{-m-1} \\ &\leq \frac{m!}{|t-x|^{m+1}} \left(\sum_{k=0}^{m} \frac{c_k}{k!(m-k)!} \langle x\rangle^{\beta-k} |t-x|^k + c_m \langle t\rangle^{\beta} \right) \\ &\leq \frac{m!}{|t-x|^{m+1}} \left(\sum_{k=0}^{m} \frac{c_k}{k!(m-k)!} \langle x\rangle^{\beta-k} 2^k \langle t\rangle^k \langle x\rangle^k + c_m \langle t\rangle^{\beta} \right) \\ &\quad \text{(Using } \langle u+v\rangle \leq 2\langle u\rangle \langle v\rangle \text{ Lemma C.0.9)} \\ &\leq \frac{m!}{|t-x|^{m+1}} \left(\langle x\rangle^{\beta} \sum_{k=0}^{m} \frac{c_k 2^k}{k!(} \langle t\rangle^k + c_m \langle t\rangle^{\beta} \right);\ x \neq t \\ &\leq d_m \langle x\rangle^{\beta-1-m} + d'_m \langle x\rangle^{-1-m} \text{ using (C.8)} \\ &\leq (d_m + d'_m) \langle x\rangle^{-1-m} \quad \text{since } \beta < 0.\end{aligned}$$

Next, the fact that $f \in C^\infty(\mathbb{R})$ implies that

> there exists a function f_m, continuous on some neighbourhood $(t-\delta_m, t+\delta_m)$, $\delta_m > 0$; of t such that

$$f_m(x) := \begin{cases} \frac{f^{(m)}(t)-f^{(m)}(x)}{t-x}, & x \in (t-\delta_m, t+\delta_m) \setminus \{t\} \\ f^{(m+1)}(t), & x = t. \end{cases}$$

From Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + \frac{1}{2}\int_x^t (t-y)^2 f'''(y)dy$$

we have

$$\frac{f(t)-f(x)}{t-x} = f'(x) + \frac{(t-x)}{2}f''(x) + \frac{1}{2(t-x)}\int_x^t (t-y)^2 f'''(y)dy.$$

Therefore

$$\begin{aligned}
\acute{f}_t^{(1)}(t) &:= \lim_{x\to t}\frac{\acute{f}_t(t)-\acute{f}_t(x)}{t-x} \\
&= \lim_{x\to t}\frac{f'(t)-\frac{f(t)-f(x)}{t-x}}{t-x} \\
&= \lim_{x\to t}\left(\frac{f'(t)-f'(x)}{t-x} - \frac{1}{2}f''(x) - \frac{1}{2(t-x)^2}\int_x^t (t-y)^2 f'''(y)dy\right) \\
&= \frac{1}{2}f''(t).
\end{aligned}$$

Inductively, $\acute{f}_t^{(m)}(t) = \frac{1}{(m+1)!}f^{(m+1)}(t)$.

Consider $[t-\epsilon, t+\epsilon] \subset (t-\delta_m, t+\delta_m)$ for some $\epsilon : 0 < \epsilon < \delta_m$. Then

1. f_m and $\acute{f}_t^{(m)}$ are continuous and bounded on $[t-\epsilon, t+\epsilon]$.

2. $(m+1)!\acute{f}_t^{(m)}(t) = f_m(t) = f^{(m+1)}(t)$.

Because of continuity of $\acute{f}_t^{(m)}$ and $f^{(m+1)}$, we can find $\rho_m \in \mathbb{R}$ such that

$$\begin{aligned} \left|\acute{f}_t^{(m)}(x)\right| &\leq \left|f^{(m+1)}(x)\right| + \rho_m, \quad \text{on } [t-\epsilon, t+\epsilon] \\ &\leq c_{m+1} \langle x\rangle^{\beta-m-1} + |\rho_m| \\ \text{which implies } \langle x\rangle^{m+1-\beta} \left|\acute{f}_t^{(m)}(x)\right| &\leq c_{m+1} + |\rho_m| \langle x\rangle^{m+1-\beta} \end{aligned}$$

Since $\langle x\rangle^{m+1-\beta}$ is continuous on $[t-\epsilon, t+\epsilon]$, it is bounded and attains its bounds there. Let $c'_{m+1} := c_{m+1} + |\rho_m| \max_{x\in[t-\epsilon,t+\epsilon]} \left\{\langle x\rangle^{m+1-\beta}\right\}$.

Then
$$\begin{aligned} \left|\acute{f}_t^{(m)}(x)\right| &\leq c'_{m+1} \langle x\rangle^{\beta-m-1} \\ &\leq c'_{m+1} \langle x\rangle^{-m-1} \text{ (since } \beta < 0 \text{ and } \langle x\rangle \geq 1) \\ & x \in [t-\epsilon, t+\epsilon]. \end{aligned}$$

Thus $\acute{f}_t \in \mathfrak{S}^{-1}$. □

3.4 Extensions of $C^\infty([0,\infty))$ functions to $\mathbb{R}$.

We next present a series of results about smooth functions initially defined on the half real line but extendible to the whole real line. In particular we wish to obtain an extension preserving the decay condition (3.1).

Lemma 3.4.1

There are sequences $\{a_k\}$, $\{b_k\}$ *such that*

1. $b_k < 0$ *for all* k.
2. $\sum_{k=0}^{\infty} |a_k| |b_k|^n < \infty \; n = 0, 1, 2, \ldots$
3. $\sum_{k=0}^{\infty} a_k (b_k)^n = 1$ *for* $n = 0, 1, 2, \ldots$
4. $b_k \to -\infty$ *as* $k \to \infty$.

Proof. see Seeley, [See64]. □

Theorem 3.4.2 (Seeley)

Let $\phi \in C^\infty(\mathbb{R})$ be such that ϕ is bounded on $\mathbb{R}$ and

$$\phi(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t \geq 2. \end{cases}$$

Define $E : C^\infty([0,\infty)) \to C^\infty(\mathbb{R})$ by

$$(Ef)(t) := \begin{cases} \sum_{k=0}^\infty a_k \phi(b_k t) f(b_k t) & , t < 0 \\ f(t) & , t \geq 0. \end{cases}$$

Where $\{a_k\}, \{b_k\}$ are the sequences described in Lemma 3.4.1.

Then E is a continuous linear extension operator.

PROOF. Again, see Seeley, [See64]. □

Lemma 3.4.3

If $f \in C^\infty(\mathbb{R}^+)$ with

$$\left|\frac{d^r}{dx^r} f(x)\right| \leq c_r \langle x \rangle^{\beta - r} \tag{3.6}$$

for some $\beta < 0$, all $r \geq 0$ and for all $x \geq 0$, then $Ef \in \mathfrak{S}^\beta \subset \mathfrak{A}$, where E is Seeley's extension operator.

PROOF. Using notations of theorem 3.4.2 and lemma 3.4.1,
first observe that $\phi^{(r-\nu)}(b_k x)$ vanishes everywhere except on the set $Q := \{x : 1 \leq b_k x \leq 2\}$. So for $x \in Q$, we have $1 \leq (b_k x)^2 \leq 4$ whence $2 \leq 1 + (b_k x)^2 \leq 5$ or equivalently $\frac{1}{\sqrt{5}} \leq \frac{1}{\langle b_k x \rangle} \leq \frac{1}{\sqrt{2}}$. So we can find a constant n_r such that $\langle b_k x \rangle^{\beta - \nu} \leq n_r \langle b_k x \rangle^{\beta - r}$, $\beta < 0$ and all $0 \leq \nu \leq r$.

Thus

$$\begin{aligned}\left|\frac{d^r}{dx^r}(Ef)(x)\right| &\leq \sum_{k=0}^{\infty} |a_k|\,|b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!}\left|\phi^{(r-\nu)}(b_k x)\right| \left|f^{(\nu)}(b_k x)\right| \\ &\leq \sum_{k=0}^{\infty} |a_k|\,|b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!}\left|\phi^{(r-\nu)}(b_k x)\right| c_\nu \langle b_k x\rangle^{\beta-\nu} \\ &\leq \sum_{k=0}^{\infty} |a_k|\,|b_k|^r M_r \sum_{\nu}^{r} c_\nu n_r \langle b_k x\rangle^{\beta-\nu} \qquad \text{for all } x<0\end{aligned}$$

where

$$\begin{aligned} M_r &:= \max_{0\leq\nu\leq r}\left\{\frac{(r+1)!}{\nu!(r-\nu)!}\sup_{x<0}\left|\phi^{(r-\nu)}(b_k x)\right|\right\} \\ &< \infty, \quad \text{since } \phi^{(m)} \text{ is bounded on } \mathbb{R} \quad \text{for all } m.\end{aligned}$$

Next, since $b_k \to -\infty$ as $k \to \infty$ we can find $\tilde{c} \in \mathbb{R}$ such that $\left\langle \frac{1}{b_k}\right\rangle \leq \frac{\tilde{c}}{\sqrt{2}}$ for all k and hence $\langle x\rangle = \left\langle \frac{1}{b_k} b_k x\right\rangle \leq \sqrt{2}\left\langle \frac{1}{b_k}\right\rangle \langle b_k x\rangle \leq \tilde{c}\,\langle b_k x\rangle$, (Lemma C.0.9).

Thus

$$\begin{aligned} \frac{1}{\langle b_k x\rangle} &\leq \frac{\tilde{c}}{\langle x\rangle} \quad \text{for all } x\in\mathbb{R} \text{ and all } b_k \\ \text{implies } \langle b_k x\rangle^{\beta-\nu} &\leq \tilde{c}^{\nu-\beta}\langle x\rangle^{\beta-\nu} \quad \text{for all } x\in\mathbb{R}, \text{ and all } k,\nu\in\mathbb{N}.\end{aligned}$$

So we can choose c_r' so that

$$r\cdot\max_{0\leq\nu\leq r}(c_\nu)n_r\langle b_k x\rangle^{\beta-r} \leq c_r'\tilde{c}^{r-\beta}\langle x\rangle^{\beta-r} \qquad \text{for all } x\in\mathbb{R}$$

and hence,

$$\begin{aligned}\left|\frac{d^r}{dx^r}(Ef)(x)\right| &\leq \sum_{k=0}^{\infty} |a_k| \, |b_k|^r \, c'_r M_r \tilde{c}^{r-\beta} \, \langle x \rangle^{\beta-r} \\ &\leq c'_r \tilde{c}^{r-\beta} \, \langle x \rangle^{\beta-r} \sum_{k=0}^{\infty} |a_k| \, |b_k|^r \\ &=: N_r \, \langle x \rangle^{\beta-r}, \quad x<0, \text{ and some } N_r > 0 \qquad (3.7)\end{aligned}$$

after summing up the series which converges by Lemma 3.4.1 .

Now set $D_r := \max\{c_r, N_r\}$ then

$$\left|\frac{d^r}{dx^r}(Ef)(x)\right| \leq D_r \, \langle x \rangle^{\beta-r}$$

for some $D_r > 0$, for all $r \geq 0$ and for all $x \in \mathbb{R}$. That is $Ef \in \mathfrak{S}^\beta \subset \mathfrak{A}$. □

Theorem 3.4.4

Let $f \in C^\infty(\mathbb{R}^+)$ satisfying (3.6) and define $\|.\|_n^+$ by

$$\|f\|_n^+ := \sum_{r=0}^{n} \int_0^\infty \left|f^{(r)}(x)\right| \, \langle x \rangle^{r-1} \, dx.$$

Then

$$\|Ef\|_n \leq c_n \, \|f\|_n^+$$

for some $c_n > 0$ (where E is Seeley's extension operator defined in Theorem 3.4.2).

PROOF.

$$\|Ef\|_n = \sum_{r=0}^{n} \left\{ \int_0^\infty \left|f^{(r)}(x)\right| \, \langle x \rangle^{r-1} \, dx + \int_{-\infty}^{0} \left|F^{(r)}(x)\right| \, \langle x \rangle^{r-1} \, dx \right\}$$

where

$$F(x) := \sum_{k=0}^{\infty} a_k \phi(b_k x) f(b_k x).$$

Therefore

$$\begin{aligned} F^{(r)}(x) &= \sum_{k=0}^{\infty} a_k \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \frac{d^{r-\nu}}{dx^{r-\nu}} \phi(b_k x) \frac{d^{\nu}}{dx^{\nu}} f(b_k x) \\ &= \sum_{k=0}^{\infty} a_k \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} b_k^{r-\nu} \phi^{(r-\nu)}(b_k x) b_k^{\nu} f^{(\nu)}(b_k x) \\ &= \sum_{k=0}^{\infty} a_k b_k^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \phi^{(r-\nu)}(b_k x) f^{(\nu)}(b_k x). \end{aligned}$$

Also,

$$\begin{aligned} \langle x \rangle^{r-1}\, dx &\leq -\frac{\langle b_k x \rangle^{r-1}}{|b_k|^{r-1}} \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} \frac{1}{-b_k} d(b_k x) \\ &= \frac{\langle b_k x \rangle^{r-1}}{|b_k|^{r}} \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} d(b_k x) \end{aligned}$$

using Lemma C.0.11, in the Appendix C and Lemma 3.4.1[(2)]. Thus

$$\begin{aligned} \|Ef\|_n &= \|f\|_n^+ + \sum_{r=0}^{n} \left\{ \int_{-\infty}^{0} \left| F^{(r)}(x) \right| \langle x \rangle^{r-1}\, dx \right\} \\ &\leq \|f\|_n^+ + \sum_{r=0}^{n} \sum_{k=0}^{\infty} |a_k|\, |b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \int_{-\infty}^{0} \left| \phi^{(r-\nu)}(b_k x) \right| \times \\ &\qquad \times \left| f^{(\nu)}(b_k x) \right| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r} \frac{\langle b_k x \rangle^{r-1}}{|b_k|^{r}} d(b_k x) \\ &\leq \|f\|_n^+ + \sum_{r=0}^{n} M_r \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} \sum_{\nu=0}^{r} \int_{0}^{\infty} \left| f^{(\nu)}(t) \right| \langle t \rangle^{\nu-1}\, dt \end{aligned}$$

$$\text{(where } M_r := \max_{0\leq \nu \leq r} \left\{ \frac{r!}{\nu!(r-\nu)!} \sup_{x<0} \left| \phi^{(r-\nu)}(b_k x) \langle b_k x \rangle^{r-\nu} \right| \right\} < \infty,$$

$$\text{since } \phi^{(m)}(b_k x) \langle b_k x \rangle^{m} = 0 \quad \text{for all } m \text{ and all } x : b_k x > 2).$$

That is

$$\begin{aligned} \|Ef\|_n &\leq \|f\|_n^+ + \sum_{r=0}^{n} M_r \|f\|_r^+ \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^r \\ &\leq \|f\|_n^+ (1+n) L_n \|f\|_n^+ \\ \text{where} \quad L_n &:= \max_{0 \leq r \leq n} \left(M_r \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^r . \right) \end{aligned}$$

But

$$\begin{aligned} \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^r &= \left\langle |b_k| \left\langle \frac{1}{b_k^2} \right\rangle \right\rangle^r \\ &= |b_k|^r \left\langle \frac{1}{b_k^2} \right\rangle^r \left\langle \frac{1}{|b_k| \left\langle \frac{1}{b_k^2} \right\rangle} \right\rangle^r . \end{aligned}$$

Since $b_k \to -\infty$ as $k \to \infty$, we have $\left\langle \frac{1}{b_k^2} \right\rangle \to 1$ as $k \to \infty$ and $\frac{1}{\left\langle \frac{1}{b_k^2} \right\rangle} \leq 1$ for any k. Therefore we can find a constant $N_r > 0$ such that $\left\langle \frac{1}{b_k^2} \right\rangle^r \left\langle \frac{1}{|b_k| \left\langle \frac{1}{b_k^2} \right\rangle} \right\rangle^r \leq N_r$ for all k and hence

$$\begin{aligned} L_n &\leq \max_{0 \leq r \leq n} \left(M_r N_r \sum_{k=0}^{\infty} |a_k| \, |b_k|^r \right) \\ (&< \infty \qquad \text{by Lemma 3.4.1.}) \end{aligned}$$

So,

$$\|Ef\|_n \leq c_n \|f\|_n^+ \tag{3.8}$$

with $c_n = 1 + (n+1) L_n$. □

Example 3.4.5

Let $f(x) := e^{-x^n t}$, $t > 0$, integer $n \geq 1$. Then $Ef \in \mathfrak{A}$, where E is Seeley's extension operator.

Indeed,

$$f^{(r)}(x) \;\to\; u_r < \infty \; as \; x \;\to\; 0 \quad \text{for all } \; r \geq 0. \tag{3.9}$$

Thus by Theorem 3.4.2 $Ef \in C^\infty(\mathbb{R})$.

Further,

$$f^{(r)}(x) = \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k x^{nk-r} f(x), \quad r \geq 1$$

where $e_{r,k}(n) \in \mathbb{Z}$ is defined in the Appendix A. (See proposition A.0.1 and remark A.0.2).

Therefore for $x > 1$, and $r \geq 1$

$$\begin{aligned} |f^{(r)}(x)| &= \sum_{k=1}^{r} e_{r,k}(n) t^k \left|x^{nk-r}\right| |f(x)| \\ &\leq c_r\, |x|^{nr-r}\, |f(x)| \sum_{k=1}^{r} t^k \\ &= c_r\, |x|^{nr-r}\, |f(x)|\, t^r \sum_{k=0}^{r-1} \frac{1}{t^k} \end{aligned} \tag{3.10}$$

with $c_r \;:=\; \max\limits_{1 \leq k \leq r} \{e_{r,k}(n)\}$

Also by means of Taylor series expansion,

$$|f(x)| = \left|e^{-x^n t}\right| \leq \frac{(r+1)!}{t^{r+1}\,|x|^{nr+n}} \qquad x > 0, \quad r \geq 0. \tag{3.11}$$

Substituting (3.11) into (3.10) and using $\frac{1}{|x|} = \frac{\langle 1/x \rangle}{\langle x \rangle} \leq \frac{\sqrt{2}}{\langle x \rangle}$ for $x > 1$ we get,

$$\begin{aligned}
\left|f^{(r)}(x)\right| &\leq c_r\,|x|^{nr-r}\,\frac{(r+1)!t^r \sum_{k=0}^{r-1} t^{-k}}{t^{r+1}\,|x|^{nr+n}} \\
&= c_r \frac{(r+1)! \sum_{k=0}^{r-1} t^{-k}}{t}\,|x|^{-n-r}, \quad x > 1 \\
&\leq c_r \frac{(r+1)! \sum_{k=0}^{r-1} t^{-k} \left(\sqrt{2}\right)^{n+r}}{t} \langle x \rangle^{-n-r} \\
&=: d_r \langle x \rangle^{-n-r}, \quad r \geq 1.
\end{aligned}$$

From (3.11) and comments following it we can set $d_0 := \frac{(\sqrt{2})^n}{t}$.

For the case $x \leq 1$, since $f^{(r)}(x)$ is bounded on $[0,1]$ for all $r \geq 0$,

$$\begin{aligned}
\left|f^{(r)}(x)\right| &\leq \sup_{x \in [0,1]} \left|f^{(r)}(x)\right| \\
&=: \left|f^{(r)}\right|_I \\
&< \infty \\
(\text{with } I &:= [0,1]).
\end{aligned}$$

But then we can find a constant $M_r' > 0$ such that

$$\left|f^{(r)}\right|_I \leq d_r M_r' \langle x \rangle^{-n-r}, \qquad x \in I$$

since $1 \leq \langle x \rangle \leq \sqrt{2}$ for $x \in [0,1]$. Now set

$$p_r := d_r \max\{1, M_r'\}, \quad r \geq 0 \tag{3.12}$$

whence

$$\left|f^{(r)}(x)\right| \leq p_r \langle x \rangle^{-n-r} \quad \text{for all } x \in [0,\infty); \quad r \geq 0. \tag{3.13}$$

Thus by Lemma 3.4.3, $Ef \in \mathfrak{S}^{-n} \subset \mathfrak{A}$. □

REMARK 3.4.6

Note that if $t \geq 1$, then the constant d_r (and hence p_r), does not depend on t, since in this case

$$\begin{aligned} d_r &= c_r \frac{(r+1)!\left(\sqrt{2}\right)^{n+r} \sum_{k=0}^{r-1} t^{-k}}{t} \\ &\leq c_r \frac{r(r+1)!\left(\sqrt{2}\right)^{n+r}}{t} \\ &\leq c_r r(r+1)!\left(\sqrt{2}\right)^{n+r} \\ &=: d_r'. \end{aligned}$$

Notes and remarks on Chapter 3

1. The following are known results used in this chapter

   Lemma 3.3.2 Lemma 3.4.1 Theorem 3.4.1
   Theorem 3.4.2

2. Our contributions in this chapter include:

   Lemma 3.2.2 Lemma 3.2.5 Lemma 3.3.5
   Lemma 3.4.3 Theorem 3.2.3 Theorem 3.3.6
   Theorem 3.4.4 Example 3.3.3 Example 3.4.5
   Corollary 3.3.4

Chapter 4

The functional calculus

4.1 Pre-requisites

In this chapter and the subsequent chapters we will need some notations and theorems from the theory of functions of complex variables. Proofs of theorems in this section can be found in Conway [Con95] and are therefore omitted.

Definition 4.1.1

If γ is a rectifiable Jordan curve and $n(\gamma : a)$ denotes the **winding number** for $a \in \mathbb{C} \setminus \gamma$ then γ is said to be **positively oriented** if $n(\gamma : a) = 1$. A curve γ is **smooth** if it is continuously differentiable and $\gamma'(t) \neq 0$ for all t. A **positive Jordan system** is a collection $\Gamma := \{\gamma_1, \ldots, \gamma_m\}$ of pairwise disjoint rectifiable Jordan curves such that for all $a \notin \gamma_j$, for all j

$$n(\Gamma : a) := \sum_{j=1}^{m} n(\gamma_j : a) = 0 \textit{ or } 1$$

The **outside of** Γ is the set **out** $\Gamma := \{a \in \mathbb{C} \ : \ n(\Gamma : a) = 0\}$ and the

inside of of Γ is the set **ins** $\Gamma := \{a \in \mathbb{C} \, : \, n(\Gamma : a) = 1\}$.

We shall use the same notation for the trace of a curve and the curve itself. What is meant will be clear from the context.

The following lemma will be used a lot in our proofs.

Lemma 4.1.2

If G is an open subset of $\mathbb{C}$ and $K \subseteq G$ is compact, then there exists a smooth, positively oriented Jordan system $\Gamma := \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\}$ contained in G such that $K \subseteq \text{ins}\,\Gamma \subseteq G$.

PROOF. See [Con95, page 4]. □

It is often convenient to think of functions f defined on $\mathbb{C}$ as functions of the complex variables z and $\bar{z}$ rather than the real variables x and y. These two sets of variables are related by the formulas

$$\begin{aligned} z &= x + iy & \bar{z} &= x - iy \\ x &= \tfrac{z+\bar{z}}{2} & y &= \tfrac{z-\bar{z}}{2i} \end{aligned}$$

Therefore if f is differentiable on an open set G, we define the derivative of f with respect to z and $\bar{z}$ by

$$\begin{aligned} \partial f &= \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \\ \bar{\partial} f &= \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \end{aligned}$$

These formulas can be justified by an application of the chain rule.

Integrals with respect to area measures on $\mathbb{C}$ will be denoted by

$$\int_G f dxdy = \int_G f(z)dxdy = \int_G f(x,y)dxdy \qquad \text{with } z =: x + iy.$$

Contour integrals will be denoted by $\int_\Gamma f dz = \int_\Gamma f(z)dz$.

Often we will switch from area to contour integration with the help of Green's Theorem,

Theorem 4.1.3 (Green's Theorem)

If Γ is a smooth positive Jordan system with $G :=$ ins Γ, $f \in C(\bar{G}) \cap C^1(G)$ and $\frac{\partial f}{\partial \bar{z}}$ is integrable over G then

$$\int_\Gamma f(z)dz = 2i \int_G \frac{\partial f}{\partial \bar{z}} dxdy. \tag{4.1}$$

Corollary 4.1.4

Let H be an operator on $\mathcal{X}$ with $G \subset \rho(H)$ and $g(z) := f(z)(z-H)^{-1}$ is such that $g \in C(\bar{G}) \cap C^1(G)$ and $\frac{\partial g}{\partial \bar{z}}$ is integrable over G then

$$\int_G \frac{\partial}{\partial \bar{z}} f(z)(z-H)^{-1} dxdy = \frac{1}{2i} \int_\Gamma f(z)(z-H)^{-1} dz. \tag{4.2}$$

PROOF.

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} \left(f(z)(z-H)^{-1} \right) &= \frac{\partial}{\partial \bar{z}} f(z) \cdot (z-H)^{-1} + \frac{\partial}{\partial \bar{z}} (z-H)^{-1} \cdot f(z) \\ &= \frac{\partial}{\partial \bar{z}} f(z) \cdot (z-H)^{-1} - (z-H)^{-2} \cdot f(z) \frac{\partial z}{\partial \bar{z}}. \end{aligned}$$

Using the formula $\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right)$, put $g(z) = z := x + iy$.
Then $\frac{\partial z}{\partial \bar{z}} = \frac{1}{2}(1-1) = 0$.
Therefore $\frac{\partial}{\partial \bar{z}} \left(f(z)(z-H)^{-1} \right) = \frac{\partial}{\partial \bar{z}} f(z) \cdot (z-H)^{-1}$. (4.2) now follows from (4.1). □

Theorem 4.1.5 (Cauchy-Green formula)

If $f \in C_c^1(\mathbb{C})$ and $w \in \mathbb{C}$ then

$$f(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z-w)^{-1}dxdy. \tag{4.3}$$

4.2 The definition

The motivation for our definition of $f(H)$, for H of $(\alpha,\ \alpha)-type\ \mathbb{R}$ and $f \in \mathfrak{A}$, comes from two ideas. Firstly, a version of Hörmander's concept of almost analytic extensions [Hor70, Hor83], as contained in the following definition.

Definition 4.2.1

Given $f \in \mathfrak{A}$ and $n \geq 0$, an **almost analytic extension** of f to $\mathbb{C}$ is

$$\begin{aligned}\tilde{f}_{\varphi,n}(x,y) &:= \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\varphi(x,y) \\ &:= \left\{ f(x) + \sum_{r=1}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right\} \varphi(x,y)\end{aligned} \tag{4.4}$$

where

$$\varphi(x,y) = \tau\left(\frac{y}{\langle x\rangle}\right) \tag{4.5}$$

and τ is non-negative $C_c^\infty(\mathbb{R})$ function such that $\tau(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2. \end{cases}$

Lemma 4.2.2

Let $f \in \mathfrak{A}$, then $\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| = O(|y|^n)$ as $|y| \to 0$ for a fixed x. Moreover we can find $c' \in \mathbb{R}$ such that $\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| \leq c'|y|^n$ as $z \to x \in \mathbb{R}$.

PROOF.

$$\frac{\partial}{\partial x}(\tilde{f}_{\varphi,n}(z)) = \sum_{r=0}^{n} \frac{f^{(r+1)}(x)(iy)^r}{r!}\varphi(x,y) + \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\varphi_x(x,y)$$

and

$$\frac{\partial}{\partial y}(\tilde{f}_{\varphi,n}(z)) = \sum_{r=1}^{n} \frac{f^{(r)}(x)i(iy)^{r-1}}{(r-1)!}\varphi(x,y) + \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\varphi_y(x,y)$$

$$\begin{aligned} \text{thus } \frac{\partial}{\partial \bar{z}}(\tilde{f}_{\varphi,n}(z)) &= \frac{1}{2}\left(\frac{\partial \tilde{f}_{\varphi,n}}{\partial x} + i\frac{\partial \tilde{f}_{\varphi,n}}{\partial y}\right)(z) \\ &= \frac{1}{2}\sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}(\varphi_x + i\varphi_y)(x,y) + \\ &+ \frac{1}{2}\left[\sum_{r=1}^{n} \frac{f^{(r)}(x)(iy)^{r-1}}{(r-1)!} + \frac{f^{(n+1)}(x)(iy)^n}{n!} - \sum_{r=1}^{n} \frac{f^{(r)}(x)(iy)^{r-1}}{(r-1)!}\right]\varphi(x,y) \\ &= \frac{1}{2}\left(\sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\right)(\varphi_x + i\varphi_y)(z) + \frac{1}{2}f^{(n+1)}(x)\frac{(iy)^n}{n!}\varphi(z) \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned} \operatorname{supp}(\varphi_x + i\varphi_y) &\subseteq \left\{(x,y) : 1 \leq \frac{|y|}{\langle x \rangle} \leq 2\right\} \\ &= \{(x,y) : \langle x \rangle \leq |y| \leq 2\langle x \rangle\} \\ &\subset \{(x,y) : 1 \leq |y| \leq 2\langle x \rangle\}. \end{aligned} \tag{4.7}$$

Therefore $\varphi_x + i\varphi_y$ vanishes on the strip $\Omega := \{(x,y) : -1 \leq y \leq 1\}$. So

$$\begin{aligned} \left|\frac{\partial}{\partial \bar{z}}(\tilde{f}_{\varphi,n}(x,y))\right| &= \frac{1}{2}\left|f^{(n+1)}(x)\right|\frac{|y|^n}{n!} \quad \text{for } (x,y) \in \Omega \\ &= M_x|y|^n \end{aligned}$$

With $M_x = \frac{|f^{(n+1)}(x)|}{2n!}$. Thus, $\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| = O(|y|^n)$ *as* $|y| \to 0$ for a fixed x.

Moreover, since $f \in \mathfrak{A}$ we can find some $\beta < 0$ and $c' > 0$ such that

$$\begin{aligned} M_x &= \frac{|f^{(n+1)}(x)|}{2n!} \\ &\leq c' \langle x \rangle^{\beta-n-1} \quad \text{for all } (x,y) \in \Omega \\ &\leq c' \\ &< \infty \end{aligned}$$

since $\langle x \rangle^{\beta-n-1} \leq 1$ for all $x \in \mathbb{R}$. Therefore $\left|\frac{\partial}{\partial \bar{z}}(\tilde{f}_{\varphi,n}(x,y))\right| \leq c' |y|^n$ as $z \to x \in \mathbb{R}$. □

Lemma 4.2.3

If $\varphi(x,y) := \tau\left(\frac{y}{\langle x \rangle}\right)$ with τ, a non-negative $C_c^\infty(\mathbb{R})$ function such that

$$\tau(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2, \end{cases}$$

then $|(\varphi_x + i\varphi_y)(x,y)| \leq \frac{K}{\langle x \rangle}$ for some $K > 0$.

Proof.

$$\begin{aligned} |\varphi_x(x,y)| &= \left|\frac{\partial}{\partial x}\tau\left(\frac{y}{\langle x \rangle}\right)\right| \\ &= \left|\tau'\left(\frac{y}{\langle x \rangle}\right) \cdot \frac{\partial}{\partial x}(y \langle x \rangle^{-1})\right| \\ &\leq \left|-y\tau'\left(\frac{y}{\langle x \rangle}\right) \langle x \rangle^{-2}\right| \quad \text{(by Lemma } C.0.12\text{)}. \end{aligned}$$

Also

$$\begin{aligned} \varphi_y(x,y) &= \frac{\partial}{\partial y}\tau\left(\frac{y}{\langle x \rangle}\right) \\ &= \tau'\left(\frac{y}{\langle x \rangle}\right) \cdot \frac{\partial}{\partial y}(y \langle x \rangle^{-1}) \\ &= \tau'\left(\frac{y}{\langle x \rangle}\right) \langle x \rangle^{-1}. \end{aligned}$$

Therefore, since τ' is bounded on $\mathbb{R}$, we can set $K := 3\sup_{s\in\mathbb{R}} |\tau'(s)|$ to obtain,

$$\begin{aligned}
|(\varphi_x + i\varphi_y)(x,y)| &\leq \frac{K}{3}\left[\frac{|y|}{\langle x\rangle^2} + \frac{1}{\langle x\rangle}\right] \\
&\leq \frac{K}{3}\left[\frac{2\langle x\rangle}{\langle x\rangle^2} + \frac{1}{\langle x\rangle}\right] \quad \text{(using (4.7))} \\
&\leq \frac{K}{\langle x\rangle}.
\end{aligned}$$

□

The second idea in our definition of $f(H)$ comes from the Helffer and Sjöstrand [HS89] integral formula,

$$f(H) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z-H)^{-1}dxdy \tag{4.8}$$

for a suitable function and operator H.

Theorem 4.2.4

Let $n > \alpha \geq 0$, $f \in \mathfrak{A}$ and H be of $(\alpha, \alpha+1)$ – type $\mathbb{R}$. Then the integral

$$f(H) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}dxdy \tag{4.9}$$

is norm convergent and defines an operator in $\mathfrak{B}(\mathcal{X})$ with

$$\|f(H)\| \leq c_\alpha \|f\|_{n+1} \quad \text{for some } c_\alpha > 0. \tag{4.10}$$

Proof. Suppose $\|(z-H)^{-1}\| \leq c\frac{\langle z\rangle^\alpha}{|\Im z|^{\alpha+1}}$ for all $z \notin \mathbb{R}$ (hypothesis). We will use the notation $(x,y) = x + iy := z$.

We observe that by (4.6), $\frac{\partial \tilde{f}}{\partial \bar{z}}$ is continuous and hence the integrand is norm continuous for $z \notin \mathbb{R}$.

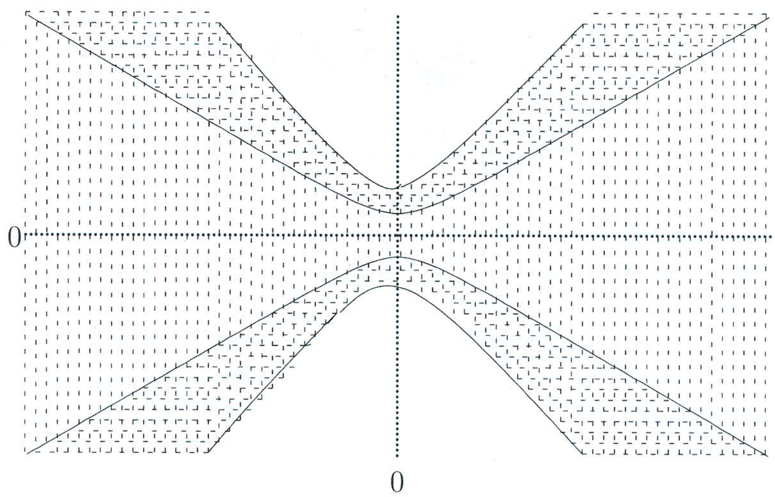

Figure 4.1: Supports of φ and $\varphi_x + i\varphi_y$

Further,

1.
$$\begin{aligned}\operatorname{supp}(\varphi) &\subseteq \left\{(x,y) : \tau\left(\frac{y}{\langle x\rangle}\right) > 0\right\} \\ &\subseteq \left\{(x,y) : \frac{|y|}{\langle x\rangle} \leq 2\right\} \\ &= \{(x,y) : 0 \leq |y| \leq 2\langle x\rangle\} \\ &=: V\end{aligned}$$

2.
$$\begin{aligned}\operatorname{supp}(\varphi_x + i\varphi_y) &\subseteq \left\{(x,y) : 1 \leq \frac{|y|}{\langle x\rangle} \leq 2\right\} \\ &= \{(x,y) : \langle x\rangle \leq |y| \leq 2\langle x\rangle\} \\ &=: U.\end{aligned}$$

3. For $z \in [\mathrm{supp}\,(\varphi) \cup \mathrm{supp}\,(\varphi_x + \varphi_y)] \setminus \mathbb{R}$,

$$\begin{aligned}
\left\|(z-H)^{-1}\right\| &\leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} \\
&= c\frac{\left(1+|x|^2+|y|^2\right)^{\alpha/2}}{|y|^{\alpha+1}} \\
&\leq c\frac{\left(1+|x|^2+4\langle x\rangle^2\right)^{\alpha/2}}{|y|^{\alpha+1}} \\
&\leq c5^{\alpha/2}\frac{\langle x\rangle^{\alpha}}{|y|^{\alpha+1}}.
\end{aligned}$$

4. $|(\varphi_x + i\varphi_y)(x,y)| \leq \frac{K}{\langle x\rangle}$ for some $K > 0$, Lemma 4.2.3.

   Also, since φ is bounded, let $M := \sup_{z\in\mathbb{C}}\{|\varphi(z)|\}$.

Therefore, using the expansion (4.6) and the estimates above, we have

$$\begin{aligned}
\|f(H)\| &\leq \frac{c5^{\frac{\alpha}{2}}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}\left|f^{(r)}(x)\right||y|^r\frac{K}{\langle x\rangle}\chi_U(z) + M\left|f^{(n+1)}(x)\right||y|^n\chi_V(z)\right)\frac{\langle x\rangle^{\alpha}}{|y|^{\alpha+1}}dxdy \\
&= \frac{c5^{\frac{\alpha}{2}}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}|f^r(x)|\,|y|^{r-\alpha-1}K\langle x\rangle^{\alpha-1}\chi_U(z)+\right. \\
&\qquad\left.+M\left|f^{(n+1)}(x)\right||y|^{n-\alpha-1}\langle x\rangle^{\alpha}\chi_V(z)\right)dxdy.
\end{aligned}$$

Integrating with respect to y yields the bound

$$\begin{aligned}
\|f(H)\| &\leq \frac{c'5^{\frac{\alpha}{2}}}{\pi}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}|f^r(x)|\left[|y|^{r-\alpha}\right]_{\langle x\rangle}^{2\langle x\rangle}\cdot\langle x\rangle^{\alpha-1} + \left|f^{(n+1)}(x)\right|\left[|y|^{n-\alpha}\right]_0^{2\langle x\rangle}\cdot\langle x\rangle^{\alpha}\right)dx \\
&= c_\alpha\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}\left|f^{(r)}(x)\right|\langle x\rangle^{r-1} + \left|f^{(n+1)}\right|\langle x\rangle^n\right)dx \\
&= c_\alpha\|f\|_{n+1} \qquad \text{with } c_\alpha := \frac{c5^{\alpha/2}2^{n-\alpha}}{\pi}\cdot\max\{K, M\}.
\end{aligned}$$

□

For an operator H of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$, we can now associate each element f of $\mathfrak{A}$ with an operator $f(H) \in \mathfrak{B}(\mathcal{X})$ given by the formula (4.9).

We then study the properties of the map

$$\begin{aligned} \kappa : \mathfrak{A} &\rightarrow \mathfrak{B}(\mathcal{X}) \\ f &\mapsto f(H). \end{aligned}$$

REMARK 4.2.5

1. The integral (4.8) appears in Helffer and Sjöstrand [HS89], see note 1 at the end of this chapter. Similar integrals play a central role in the theory of uniform algebras, Gamelin [Gam69].

2. It may seem from the computation above that our definition of $f(H)$ depends implicitly on the cut-off function φ and n. However we will prove shortly that $f(H)$ is independent of both φ and n, provided $n > \alpha$.

Lemma 4.2.6

If $F \in C_c^\infty(C)$ and $F(z) = O(|y|^\beta)$ as $y \rightarrow 0$ for some $\beta > \alpha + 1$, then

$$-\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}(z - H)^{-1} dxdy = 0. \tag{4.11}$$

PROOF. Let F have support in $\{z = (x, y) : |x| < N \text{ and } |y| < N\}$ and define Ω_δ for small $\delta > 0$ to be the region $\{z = (x, y) : |x| < N \text{ and } \delta < |y| < N\}$ (see figure 4.2).

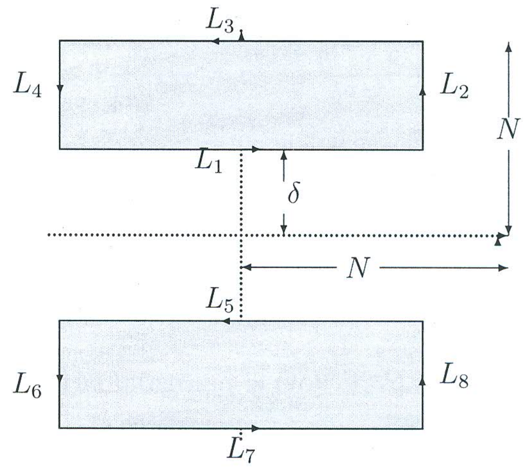

Figure 4.2: Close up on the support of F by compact regions.

$$\begin{aligned}
A &:= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}(z-H)^{-1}dxdy \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}(z-H)^{-1}dxdy \\
&= -\lim_{\delta \to 0}\frac{1}{\pi}\int_{\Omega_\delta} \frac{\partial F}{\partial \bar{z}}(z-H)^{-1}dxdy \\
&= \lim_{\delta \to 0}\frac{i}{2\pi}\int_{\partial\Omega_\delta} F(z)(z-H)^{-1}dz \quad \text{(Green's Theorem)} \\
&= \lim_{\delta \to 0}\frac{i}{2\pi}\sum_{r=1}^{8}\int_{L_r} F(z)(z-H)^{-1}dz \\
&= \lim_{\delta \to 0}\frac{i}{2\pi}\left(\int_{L_1} F(z)(z-H)^{-1}dz + \int_{L_5} F(z)(z-H)^{-1}dz\right) \\
&\quad \text{since } [\operatorname{supp}(F)] \cap \left[\left(\cup_{r=2}^{4} L_r\right) \cup \left(\cup_{r=6}^{8} L_r\right)\right] = \emptyset.
\end{aligned}$$

Now for $(x,y) \in L_1 \cup L_5 \subset \mathbb{C} \setminus \mathbb{R}$,

$$\|(z-H)^{-1}\| \leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} = c\frac{(1+|x|^2+\delta^2)^{\alpha/2}}{\delta^{\alpha+1}} \leq c\frac{2^{\alpha/2}\langle N\rangle^{\alpha}}{\delta^{\alpha+1}}.$$

Therefore

$$\|A\| \leq c2^{\alpha/2}\langle N\rangle^{\alpha} \lim_{\delta\to 0}\int_{-N}^{N}\{|F(x+i\delta)|+|F(x-i\delta)|\}\delta^{-\alpha-1}dx = 0,$$

since by hypothesis the integrand is $O(\delta^{\beta-\alpha-1})$. □

Theorem 4.2.7

The operator $f(H)$ is independent of n and the cut-off function φ defined in (4.5), provided $n > \alpha$.

PROOF. The norm density result of Lemma 3.2.5 together with (4.10) imply that it is sufficient to prove this for $f \in C_c^{\infty}$.

If $f \in C_c^{\infty}(\mathbb{R})$ while φ_1 and φ_2 are cut-off functions define in terms of say τ_1 and τ_2, let

$$\begin{aligned}\Omega_1 &:= \left\{(x,y) \,:\, \frac{|y|}{\langle x\rangle} < \epsilon_1\right\} \quad \text{for some } \epsilon_1 > 0\\ &= \{(x,y) \,:\, -\epsilon_1\langle x\rangle < y < \epsilon_1\langle x\rangle\}\\ &\subseteq \{z \,:\, \varphi_1(z) = 1\}.\end{aligned}$$

Similarly let

$$\begin{aligned}\Omega_2 &:= \{(x,y) \,:\, -\epsilon_2\langle x\rangle < y < \epsilon_2\langle x\rangle\} \quad \text{for some } \epsilon_2 > 0\\ &\subseteq \{z \,:\, \varphi_2(z) = 1\}.\end{aligned}$$

$$\begin{aligned}\text{Now set}\quad \Omega &:= \Omega_1 \cap \Omega_2\\ &= \{(x,y) \,:\, -\epsilon\langle x\rangle < y < \epsilon\langle x\rangle\} \quad \text{with } \epsilon := \min\{\epsilon_1,\epsilon_2\}\\ &\neq \emptyset.\end{aligned}$$

Then for $z \in \Omega$,

$$\begin{aligned} \tilde{f}_{\varphi_1,n}(z) - \tilde{f}_{\varphi_2,n}(z) &= \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}[\varphi_1(z) - \varphi_2(z)] \\ &= 0 \quad \text{since } \varphi_1(z) = \varphi_2(z) = 1 \text{ for all } \; z \in \Omega. \end{aligned}$$

This exceeds the hypothesis of lemma 4.2.6, so invoking lemma 4.2.6, we have $\tilde{f}_{\varphi_1,n}(H) = \tilde{f}_{\varphi_2,n}(H)$. That is $\tilde{f}_{\varphi,n}(H)$ is independent of φ.

On the other hand, if $m > n > \alpha$ then

$$\begin{aligned} \tilde{f}_{\varphi_1,m}(z) - \tilde{f}_{\varphi_1,n}(z) &= \left(\sum_{r=0}^{m} \frac{f^{(r)}(x)(iy)^r}{r!} - \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right) \varphi_1(z) \\ &= \sum_{r=n+1}^{m} \frac{f^{(r)}(x)(iy)^r}{r!} \varphi_1(z) \\ &=: \; y^{n+1} K(z) \qquad \text{(some bounded } \; K : \mathbb{C} \to \mathbb{C}) \end{aligned}$$

and since $n + 1 > \alpha + 1$ we invoke Lemma 4.2.6 to conclude that $\tilde{f}_{\varphi_1,m}(H) = \tilde{f}_{\varphi_1,n}(H)$. That is $\tilde{f}_{\varphi,n}(H)$ is independent of n. □

4.3 The homomorphism $\mathfrak{A} \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{X})$

Henceforth we will assume that the condition of theorem 4.2.7 holds and write $\tilde{f}$ instead of $\tilde{f}_{\varphi,n}$ unless a specific cut-off function or n is needed for some purpose which will be stated.

Theorem 4.3.1

Let H be an operator of $(\alpha, \; \alpha + 1) - type \; \mathbb{R}$ for some $\alpha \geq 0$. If $f \in C_c^{\infty}(\mathbb{R})$ has support disjoint from $\sigma(H)$, then $f(H) = 0$.

Proof. By definition 3.2.4, supp $\left(\tilde{f}\right)$ is a closed set. Thus by regularity of $\mathbb{C}$, we can find an open set G with $\operatorname{supp}\left(\tilde{f}\right) \subset G$ and $G \cap \sigma(H) = \emptyset$. Since by hypothesis, $\operatorname{supp}\left(\tilde{f}\right)$ is compact, there exists a finite set of smooth curves, $\{\Upsilon_r\}_{r=1}^m$ 'enclosing' $\operatorname{supp}\left(\tilde{f}\right)$ in G, Lemma 4.1.2. Thus if we put $\Gamma := \cup_{r=1}^m \Upsilon_r$, and $D := ins\Gamma$ then

$$\begin{aligned}
f(H) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}}(z-H)^{-1}dz \\
&= -\frac{1}{\pi}\int_{D} \frac{\partial \tilde{f}(z)}{\partial \bar{z}}(z-H)^{-1}dz \\
&= \frac{i}{2\pi}\int_{\Gamma} \tilde{f}(z)(z-H)^{-1}dz \quad \text{(Green's Theorem)} \\
&= \frac{i}{2\pi}\sum_{r=1}^{m}\int_{\Upsilon_r} \tilde{f}(z)(z-H)^{-1}dz \\
&= 0 \quad \text{since } \tilde{f}(z) = 0 \ \text{ for all } \ z \in \Upsilon_r \ r = 1, 2, \ldots, m
\end{aligned}$$

□

Corollary 4.3.2

Let H be an operator of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ for some $\alpha \geq 0$. If $f \in \mathfrak{A}$ has support disjoint from $\sigma(H)$ then $f(H) = 0$.

PROOF. Follows from theorem 4.3.1, inequality (4.10) and lemma 3.2.5.

□

Theorem 4.3.3

If $f, g \in \mathfrak{A}$ and H is of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ then

$$(fg)(H) = f(H)g(H).$$

PROOF. We first assume that f and g lie in $C_c^\infty(\mathbb{R})$. Let $K := \operatorname{supp}\left(\tilde{f}\right)$ and $L := \operatorname{supp}(\tilde{g})$ so that K and L are compact subsets of $\mathbb{C}$ and write

$$f(H) \quad = \quad -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}dxdy, \qquad z =: x + iy$$

$$\text{and} \quad g(H) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{w}}(w-H)^{-1}dudv, \quad w =: u+iv$$

$$\text{Then} \quad f(H)g(H) = \frac{1}{\pi^2}\int\int_{K\times L} \frac{\partial \tilde{f}}{\partial \bar{z}}\frac{\partial \tilde{g}}{\partial \bar{w}}(z-H)^{-1}(w-H)^{-1}dxdydudv.^1$$

Using resolvent equation (1.5)

$$(z-H)^{-1}(w-H)^{-1} = (z-w)^{-1}(w-H)^{-1} - (z-w)^{-1}(z-H)^{-1},$$

we may expand $f(H)g(H)$ as

$$\begin{aligned} f(H)g(H) &= \frac{1}{\pi^2}\int\int_{K\times L} \frac{\partial \tilde{f}}{\partial \bar{z}}\frac{\partial \tilde{g}}{\partial \bar{w}}\left[(z-w)^{-1}(w-H)^{-1} - (z-w)^{-1}(z-H)^{-1}\right] dxdydudv \\ &= \frac{-1}{\pi}\int_{K\times L}\left(\frac{\partial \tilde{g}}{\partial \bar{w}}(w-H)^{-1}\frac{-1}{\pi}\int_K \frac{\partial \tilde{f}}{\partial \bar{z}}(z-w)^{-1}dxdy\right) dudv \\ &\quad -\frac{-1}{\pi}\int_{K\times L}\left(\frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}\frac{-1}{\pi}\int_L \frac{\partial \tilde{g}}{\partial \bar{w}}(z-w)^{-1}dudv\right) dxdy \end{aligned}$$

But

$$\begin{aligned} -\frac{1}{\pi}\int_K \frac{\partial \tilde{f}}{\partial \bar{z}}(z-w)^{-1}dxdy &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-w)^{-1}dxdy \\ &= \tilde{f}(w) \quad \text{(Cauchy-Green Theorem)}. \end{aligned}$$

Also

$$\begin{aligned} \frac{1}{\pi}\int_L \frac{\partial \tilde{g}}{\partial \bar{w}}(z-w)^{-1}dudv &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{g}}{\partial \bar{w}}(w-z)^{-1}dudv \\ &= \tilde{g}(z) \quad \text{(Cauchy-Green Theorem)}. \end{aligned}$$

[1] $K \times L := \{(k,l) \ : \ k \in K, l \in L\}$

These lead to the identity

$$\begin{aligned} f(H)g(H) &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial\tilde{g}(w)}{\partial\bar{w}}(w-H)^{-1}\tilde{f}(w)dudv + \frac{-1}{\pi}\int_{K\times L}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial\tilde{g}(z)}{\partial\bar{z}}(z-H)^{-1}\tilde{f}(z)dxdy + \frac{-1}{\pi}\int_{K\times L}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\left\{\tilde{f}(z)\frac{\partial\tilde{g}(z)}{\partial\bar{z}} + \frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)\right\}(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial(\tilde{f}\tilde{g})(z)}{\partial\bar{z}}(z-H)^{-1}dxdy. \end{aligned}$$

In order to prove that

$$fg(H) = -\frac{1}{\pi}\int_{K\times L}\frac{\partial(\tilde{f}\tilde{g})(z)}{\partial\bar{z}}(z-H)^{-1}dxdy,$$

we must prove that

$$-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial k(z)}{\partial\bar{z}}(z-H)^{-1}dxdy = 0,$$

where $k(z) := \tilde{f}(z)\tilde{g}(z) - \widetilde{(fg)}(z)$. Since k is of compact support and by Theorem 4.2.7 and Lemma4.2.2 may be assumed to satisfy the hypothesis of Lemma 4.2.6, this follows by invoking Lemma 4.2.6.

Finally, suppose that $f, g \in \mathfrak{A}$ and let $\phi \in C_c^\infty$ such that

$$\phi(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2 \end{cases}$$

(See Theorem B.0.6 in the appendix), set $\phi_m(s) := \phi(s/m)$ and $f_m := \phi_m f$, $g_m := \phi_m g$, and $h_m := \phi_m^2 fg$.
Then $f_m \to f$, $g_m \to g$ and $h_m \to fg$ in the norm $|||\cdot|||_p$ for some $p > \alpha$.

See proof of Lemma 3.2.5.

From above we have

$$h_m(H) = f_m(H)g_m(H) \quad \text{for all } \ m.$$

We finally invoke Lemma 3.2.5 and (4.10) to complete the proof. □

Lemma 4.3.4

Let $g \in \mathfrak{A}$ with $g = 0$ on $[0, \infty)$. If H is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ and $\sigma(H) \subseteq [0, \infty)$ then $g(H) = 0$.

PROOF. For $\epsilon \in (0, \infty)$, let $H_\epsilon := \epsilon + H$. Then H_ϵ is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$, Theorem 2.2.10 . But $\sigma(H) \subset [0, \infty)$ implies that $\sigma(H_\epsilon) \subset [\epsilon, \infty)$, and since $\operatorname{supp}(g) \subseteq (\infty, 0]$ we must have $g(H_\epsilon) = 0$ by Theorem 4.3.1.

$$\begin{aligned} \text{Now} \qquad 0 \ = \ g(H_\epsilon) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{w}}\tilde{g}(w)(w - (\epsilon + H))^{-1} du dv \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\tilde{g}(z+\epsilon)(z - H)^{-1} dx dy \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\tilde{g}_\epsilon(z)(z - H)^{-1} dx dy \\ &= g_\epsilon(H) \end{aligned}$$

where $z := w - \epsilon$ and $g_\epsilon \ := \ \tau_\epsilon g \in \mathfrak{A}$ by Lemma 3.2.2.

So by (4.10)

$$\|g_\epsilon(H) - g(H)\| = \|g(H)\| \leq c_\alpha \|g_\epsilon - g\|_{n+1}, \quad \text{for some } n > \alpha,\ c_\alpha > 0 \text{ for all } \epsilon > 0.$$

Suppose $g_\epsilon \in \mathfrak{S}^{\beta_\epsilon}$ for some $\beta_\epsilon < 0$ and $\epsilon \geq 0$, where we set $g_0 := g$. Then $|g_\epsilon(x)| \leq c_{r,\epsilon} \langle x \rangle^{\beta_\epsilon - r}$ for each $x \in \mathbb{R}$.

Let $\beta := \sup\{\beta_\epsilon \ : \ \epsilon \in (0,\infty)\} < 0$ and $c := \max\limits_{0<r\leq n} \sup\limits_{\epsilon\in(0,\infty)} \{c_{r,\epsilon}\} > 0$. β and c exist and are finite, see the proof of Lemma 3.2.2. Thus

$$\left|g_\epsilon^{(r)}(x)\right| \langle x\rangle^{r-1} \leq c \langle x\rangle^{\beta-r} \langle x\rangle^{r-1} = c \langle x\rangle^{\beta-1}$$

and the function $h(x) = c\langle x\rangle^{\beta-1}$ is integrable and $\left|\frac{d^r}{dx^r}\left(g_\epsilon(x) - g(x)\right)\right| \langle x\rangle^{r-1} \leq h(x)$ for each ϵ. Therefore by dominated convergence theorem we have

$$\int_0^\infty \left|g_\epsilon^{(r)}(x) - g^{(r)}(x)\right| \langle x\rangle^{r-1} dx \to 0 \text{ as } \epsilon \to 0,$$

that is $\|g_\epsilon - g\|_{n+1} \to 0$ as $\epsilon \to 0$. Thus $\|g(H)\| = 0$. □

For $f \in C^\infty([0,\infty))$ such that $Ef \in \mathfrak{A}$ we define $f(H)$ to be $Ef(H)$ where $Ef(H)$ is given by (4.9) with appropriate condition on $\|(z-H)^{-1}\|$.

Theorem 4.3.5

If $f : [0,\infty) \to \mathbb{C}$ is such that

$$\left|\frac{d^r}{dx^r} f(x)\right| \leq c_r \langle x\rangle^{\beta-r} \qquad (3.6)$$

for some $\beta < 0$, for all $r \geq 0$ and for all $x \geq 0$; and H is of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ with $\sigma(H) \subseteq [0,\infty)$, then $f(H)$ is uniquely determined and

$$\|f(H)\| \leq k\,\|f\|_{n+1}^{+}, \quad k > 0, \quad \text{whenever } n > \alpha.$$

Proof. By Lemma 3.4.3 we observe that $Ef \in \mathfrak{A}$. So that $f(H) \equiv Ef(H)$ is defined and $f(H) \in \mathfrak{B}(\mathcal{X})$.

Moreover,

$$\begin{aligned}
\|f(H)\| &= \|(Ef)(H)\| \\
&\leq c_{n+1}\,\|Ef\|_{n+1} \quad [(4.10)] \\
&\leq c' c_{n+1}\,\|f\|_{n+1}^{+} \quad \text{(Theorem 3.4.4)} \\
&=: K\,\|f\|_{n+1}^{+}\,.
\end{aligned}$$

Finally if $g \in \mathfrak{A}$ is another extension of f, set

$$h := g - Ef \in \mathfrak{A}$$

which implies $h = 0$ on $[0, \infty)$ and thus by Lemma 4.3.4

$$h(H) = 0.$$

□

Corollary 4.3.6

Let $f, g \in C^\infty([0,\infty))$ satisfy (3.6) of Lemma 3.4.3 with H of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ and $\sigma(H) \subseteq [0,\infty)$. Then

$$(fg)(H) = f(H)g(H).$$

PROOF.

$$(Ef)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t) f(b_k t), & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

and

$$(Eg)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t) g(b_k t), & t < 0 \\ g(t), & t \geq 0 \end{cases}$$

Thus $(Eg)(t)(Ef)(t) := \begin{cases} \left(\sum_{k=0}^{\infty} a_k \phi(b_k t) g(b_k t)\right) \left(\sum_{k=0}^{\infty} a_k \phi(b_k t) f(b_k t)\right), & t < 0 \\ g(t)f(t), & t \geq 0. \end{cases}$

Clearly gf satisfies (3.6) of Lemma 3.4.3 and

$$E(gf)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t)(gf)(b_k t), & t < 0 \\ g(t)f(t), & t \geq 0. \end{cases}$$

Thus, $(Eg)(t) \cdot (Ef)(t) - E(gf)(t) = 0, \quad t \geq 0.$

Therefore by Lemma 4.3.4 $(Eg)(H) \cdot (Ef)(H) = E(gf)(H)$.

i.e. $g(H) \cdot f(H) = gf(H)$. □

REMARK 4.3.7

Theorem 4.3.3 and Corollary 4.3.6 show that the map

$$\begin{aligned} \kappa : \mathfrak{A} &\rightarrow \mathfrak{B}(\mathcal{X}) \\ f &\mapsto f(H) \end{aligned}$$

is an algebra homomorphism. We prove one more result to verify that κ is a functional calculus, a concept we will define shortly.

Theorem 4.3.8

Let H be an operator of $(\alpha,\ \alpha+1)-$ type $\mathbb{R}$ for some $\alpha \geq 0$. If $w \notin \mathbb{R}$ and $r_w(x) := (w-x)^{-1}$ for all $x \in \mathbb{R}$ then $r_w \in \mathfrak{A}$ and

$$r_w(H) = (w-H)^{-1}.$$

PROOF. We have already seen that $r_w \in \mathfrak{A}$, Example 3.3.3.

Without loss of generality, suppose that $\Im w > 0$. For large $m > 0$ define $\Omega_m := \{(x, y) : |x| < m \text{ and } \frac{\langle x\rangle}{m} < |y| < 2m\}$.

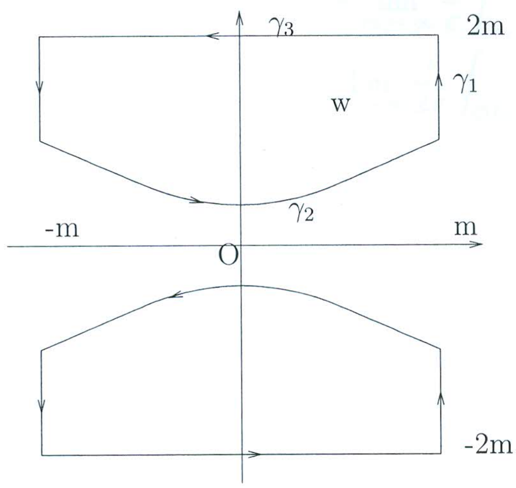

Figure 4.3: Close up on $\mathbb{C}$, over which $r_w(z)$ is integrated.

The boundary of Ω_m consists of two closed curves, both traversed in the anti-clockwise direction, see figure 4.3.

With τ as in definition 4.2.1, put

$$\varphi(z) := \tau\Big(\frac{\lambda\,|y|}{\langle x\rangle}\Big)$$

where $\lambda > 0$ is chosen

1. large enough to ensure that $w \notin \operatorname{supp}(\varphi)$.

2. so that for each $m \geq 1$, $|y| < \frac{2\langle m\rangle}{\lambda} \leq 2m$, for all $(x, y) \in \Omega_m$. Thus, $\langle 1/m\rangle \leq \lambda$. Since $\langle 1/m\rangle < 2$ for all $m \geq 1$ we may assume that $\lambda \geq 2$.

An application of Green's Theorem yields

$$\begin{aligned} r_w(H) &= -\lim_{m\to\infty}\frac{1}{\pi}\int_{\Omega_m}\frac{\partial\tilde{r}_w}{\partial\bar{z}}(z-H)^{-1}dxdy \\ &= \lim_{m\to\infty}\frac{i}{2\pi}\int_{\partial\Omega_m}\tilde{r}_w(z)(z-H)^{-1}dz. \end{aligned}$$

We next show that

$$\lim_{m\to\infty}\left\|\int_{\partial\Omega_m}\{r_w(z)-\tilde{r}_w(z)\}(z-H)^{-1}dz\right\| = 0.$$

$\partial\Omega_m$ consists of four vertical straight lines, two horizontal straight lines and two curves. The integral is estimated separately on each of these.

1. Vertical lines: Suppose γ_1 is the vertical line in the first quadrant. Using Taylor's approximation theorem to expand $r_w(z)$ at $(m,0)$, we obtain, for all $z\in\gamma_1$,

$$r_w(z) = \sum_{s=0}^{n}\frac{r_w^{(s)}(m)(iy)^s}{s!} + R(z;m)$$

with $R(z;m) := \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}$, $\quad d = m+\epsilon iy$ for some $\epsilon\in(0,1)$.

Therefore, for any $z\in\gamma_1$ we have

$$r_w(z) = \sum_{s=0}^{n}\frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}$$

which implies

$$
\begin{aligned}
&|r_w(z) - \tilde{r}_w(z)| \\
&\quad \leq \; |(1-\varphi(z))r_w(z)| + \varphi(z)\left|r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)}\right| \\
&\quad = \; c_1\chi(z)\langle z\rangle^{-1} + \varphi(z)\left|\sum_{s=0}^{n}\frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!} - \sum_{s=0}^{n} r_w^{(s)}(m)\frac{(iy)^s}{s!}\right| \\
&\quad \leq \; c_1\chi(z)\langle z\rangle^{-1} + c_w\frac{|y|^{n+1}}{\langle d\rangle^{n+2}} \quad \text{(see Example 3.3.3)}
\end{aligned}
$$

where $\chi(z) := \begin{cases} 1 & if\;\; \langle x\rangle < \lambda |y| \\ 0 & \text{otherwise.} \end{cases}$

But $z = m + iy$, $d = m + i\epsilon y$ implies $\langle z\rangle \geq \langle m\rangle$ and $\langle m\rangle \leq \langle d\rangle$. So,

$$|r_w(z) - \tilde{r}_w(z)| \leq c_1\chi(z)\langle m\rangle^{-1} + c_w\frac{|y|^{n+1}}{\langle m\rangle^{n+2}}$$

Also, for $z := m + iy \in \gamma_1$, $\frac{\langle m\rangle}{m} \leq |y| \leq 2m$ and hence

$$
\begin{aligned}
\langle z\rangle^2 &= 1 + |m|^2 + |y|^2 \\
&\leq 1 + m^2 + 4m^2 \\
&\leq 5\langle m\rangle^2 .
\end{aligned}
$$

Therefore $$\|(z-H)^{-1}\| \leq \frac{c5^{\alpha/2}\langle m\rangle^{\alpha}}{|y|^{\alpha+1}} \qquad (4.12)$$

for some $c > 0$.

Hence

$$
\begin{aligned}
&\left\| \int_{\gamma_1} \{r_w(z) - \tilde{r}_w(z)\}(z-H)^{-1}dz \right\| \\
&\leq \; cc_1 5^{\alpha/2} \int_{\frac{\langle m\rangle}{\lambda}}^{2m} \langle m\rangle^{-1} \frac{\langle m\rangle^{\alpha}}{y^{\alpha+1}} dy + cc_w 5^{\alpha/2} \int_{\frac{\langle m\rangle}{m}}^{2m} |y|^{n-\alpha} \langle m\rangle^{\alpha-n-2}\, dy \\
&\leq \; cc_1 5^{\alpha/2} \langle m\rangle^{\alpha-1} \int_{\frac{\langle m\rangle}{\lambda}}^{2m} \frac{\lambda^{\alpha+1}}{\langle m\rangle^{\alpha+1}} dy + cc_w 5^{\alpha/2} \langle m\rangle^{\alpha-n-2} \int_{\frac{\langle m\rangle}{m}}^{2m} |2m|^{n-\alpha}\, dy \\
&\leq \; cc_1 5^{\alpha/2} \langle m\rangle^{\alpha-1} \frac{\lambda^{\alpha+1}}{\langle m\rangle^{\alpha+1}} \left| 2m - \frac{\langle m\rangle}{\lambda} \right| + cc_w 5^{\alpha/2} \langle m\rangle^{\alpha-n-2} |2m|^{n-\alpha} \left| 2m - \frac{\langle m\rangle}{m} \right| \\
&= \; (m^{-1}) \left\{ c_1' \langle 1/m\rangle^{-2} \left| 2 - \frac{\langle 1/m\rangle}{\lambda} \right| + c_w' \langle 1/m\rangle^{\alpha-n-2} \left| 2 - \frac{\langle 1/m\rangle}{m} \right| \right\} \\
&= \; O(m^{-1}) \quad \text{as } m \to \infty
\end{aligned}
$$

provided $n > \alpha$. The estimate is valid for all vertical lines.

2. <u>The curves:</u> Let γ_2 be the curve in the upper half plane, i.e $\gamma_2 := \{(x,y) \; : \; y = \frac{\langle x\rangle}{m}\}$.

   Since $\frac{1}{m}\langle x\rangle < \frac{1}{\lambda}\langle x\rangle$ for all $m > \lambda$, $\varphi(z) = 1$ for all $z \in \gamma_2$ and $m > \lambda$. Therefore using Taylor's approximation at $(x,0)$ with $d := x + \epsilon i y$ for some $\epsilon \in (0,1)$ we have,

$$
\begin{aligned}
|r_w(z) - \tilde{r}_w(z)| \;&\leq\; |(1-\varphi(z))r_w(z)| + \varphi(z) \left| r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)} \right| \\
&=\; \varphi(z) \left| r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)} \right| \\
&\leq\; c_w \frac{r_w^{(n+1)}(d)\, |y|^{n+1}}{(n+2)!} \\
&\leq\; c_w \frac{|y|^{n+1}}{\langle d\rangle^{n+2}}, \quad z \in \gamma_2, \quad m > \lambda
\end{aligned}
$$

where $\langle d\rangle \geq \langle x\rangle$. Also, for $z \in \gamma_2$,

$$
\begin{aligned}
\langle z\rangle^2 &= 1 + |x|^2 + |y|^2 \\
&= \langle x\rangle^2 + \frac{\langle x\rangle^2}{m^2} \\
&= \frac{m^2+1}{m^2}\langle x\rangle^2 \\
&= \frac{\langle m\rangle^2}{m^2}\langle x\rangle^2 .
\end{aligned}
$$

Hence $\|(z-H)^{-1}\| \leq \frac{c\langle m\rangle^\alpha \langle x\rangle^\alpha}{m^\alpha |y|^{\alpha+1}}$ for some $c>0$ and

$$
\begin{aligned}
\left\|\int_{\gamma_2}\{r_w(z)-\tilde{r}_w(z)\}(z-H)^{-1}dz\right\| &\leq c_\omega c\int_{\gamma_2}\frac{|y|^{n+1}}{\langle x\rangle^{n+2}}\frac{\langle m\rangle^\alpha\langle x\rangle^\alpha}{m^\alpha |y|^{\alpha+1}}dz \\
&= c_1\int_{\gamma_2}\frac{\langle m\rangle^\alpha}{m^\alpha}\langle x\rangle^{\alpha-n-2}|y|^{n-\alpha}\,dz \\
&= c_1\int_{\gamma_2}\frac{\langle m\rangle^\alpha}{m^\alpha}\langle x\rangle^{\alpha-n-2}\frac{\langle x\rangle^{n-\alpha}}{m^{n-\alpha}}dz \\
&= c_1\frac{\langle m\rangle^\alpha}{m^n}\int_{\gamma_2}\langle x\rangle^{-2}\,dz \\
&= O(m^{\alpha-n}) \quad \text{as } m\to\infty
\end{aligned}
$$

provided $n>\alpha$. The estimate here is also valid for the other curve.

3. The horizontal lines Let γ_3 be the horizontal line in the upper half plane, i.e $\gamma_3 := \{(x,y)\ :\ y=2m\}$. Now $\operatorname{supp}(\varphi) \subset \left\{(x,y)\ :\ \frac{\lambda|y|}{\langle x\rangle}\leq 2\right\}$

   Thus $\varphi(z)=0$ for all $z\in\overline{\Omega_m}$ with $|y| > \frac{2\langle m\rangle}{\lambda}$.

   So, for $z\in\gamma_3$, $\varphi(z)=0$ if $2m > \frac{2\langle m\rangle}{\lambda}$, that is $\lambda > \langle 1/m\rangle$.

   Therefore if we choose m large enough so that $\lambda > \langle 1/m\rangle$, then for

   $z\in\gamma_3$,

$$\begin{aligned}|r_w(z) - \tilde{r}_w(z)| &= |r_w(z)| \\ &\leq \frac{c_1}{\langle z\rangle} \quad \text{using (C.8)} \\ &= \frac{c_1}{\sqrt{1+|x|^2+|2m|^2}} \\ &\leq \frac{c_1}{\langle 2m\rangle}.\end{aligned}$$

Also, for $z \in \gamma_3$,

$$\begin{aligned}\langle z\rangle^2 &= 1+|x|^2+|y|^2 \\ &\leq 1+m^2+4m^2 \\ &\leq 5\langle m\rangle^2.\end{aligned}$$

Hence $\|(z-H)^{-1}\| \quad \leq \quad \frac{c5^{\alpha/2}\langle m\rangle^\alpha}{|2m|^{\alpha+1}}$ for some $c > 0$ and

$$\begin{aligned}\left\|\int_{\gamma_3}\{r_w(z)-\tilde{r}_w(z)\}(z-H)^{-1}dz\right\| &\leq c_1 c5^{\alpha/2}\int_{\gamma_3}\frac{1}{\langle 2m\rangle}\frac{\langle m\rangle^\alpha}{(2m)^{\alpha+1}}dz \\ &\leq c_\omega m^{-2}\langle 1/2m\rangle^{-1}\langle 1/m\rangle^\alpha\int_{-m}^{m}dx \\ &= 2c_\omega m^{-2}\langle 1/2m\rangle^{-1}\langle 1/m\rangle^\alpha m \\ &= O(m^{-1}) \quad \text{as } m \to \infty\end{aligned}$$

provided $n > \alpha$. The estimate here is also valid for the other horizontal line.

Combining all the cases we obtain

$$r_w(H) = \frac{i}{2\pi}\lim_{m\to\infty}\int_{\partial\Omega_m} r_w(z)(z-H)^{-1}dz.$$

The integrand is holomorphic on and inside the part of $\partial\Omega_m$ in the lower half plane, so the contribution of that integral is zero by Cauchy's theorem. The integrand is meromorphic in the upper half plane with a single pole at $z = w$.

Therefore

$$\begin{aligned} r_w(H) &= -\mathrm{Res}_{z=w}\{r_w(z)(z-H)^{-1}\} \\ &= (w-H)^{-1} \end{aligned}$$

where $\mathrm{Res}_{z=w} f(z)$ denotes the residue of f at the pole w. □

Definition 4.3.9

By a **$\mathfrak{A}$-functional calculus** for an operator H of $(\alpha,\ \alpha+1)-$ *type* $\mathbb{R}$ we will mean a continuous linear map κ from $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{X})$ such that

1. $\kappa(fg) = \kappa(f)\kappa(g)$, for all $f, g, \in \mathfrak{A}$.

2. If $w \notin \mathbb{R}$ then $r_w \in \mathfrak{A}$ and $\kappa(r_w) = (w-H)^{-1}$ (r_w is defined in Example 3.3.3 and Theorem 4.3.8).

Note that in this definition $\kappa(f) \equiv f(H)$.

Lemma 4.3.10

Let $f \in C_0(\mathbb{R})$, H a closed operator with $\sigma(H) \subset \mathbb{R}$ and $\lambda \in \mathbb{R}\setminus\{0\}$ such that $f^{-1}(\lambda) \neq \emptyset$ and $f^{-1}(\lambda) \cap \sigma(H) = \emptyset$. Then there exists a smooth function $\phi \in C^\infty(\mathbb{R})$ and a neighbourhood G of $\sigma(H)$ such that

$$\phi(t) = \begin{cases} 0 & if\ t \in f^{-1}(\lambda) \\ 1 & if\ t \in G. \end{cases}$$

Proof. Let $x_0 \in \mathbb{R}$ be such that $f(x_0) = \lambda$ and $d := \mathrm{dist}(x_0, \sigma(H)) > 0$. Choose $\epsilon_0 \in \mathbb{R}$: $0 < \epsilon_0 < d$. Let G_0 be an open set such that

$$[x_0 - \epsilon_0, x_0 + \epsilon_0] \subset G_0 \subset [x_0 - d, x_0 + d].$$

(Using Theorem B.0.6 in the appendix), choose a smooth function ψ_0 such that

$$\psi_0(t) = \begin{cases} 1 & if\ t \in [x_0 - \epsilon_0, x_0 + \epsilon] \\ 0 & if\ t \in \mathbb{R} \setminus G_0. \end{cases}$$

Next, set $\phi_0 := 1 - \psi_0$, then clearly ϕ_0 is smooth and

$$\phi_0(t) = \begin{cases} 0 & if\ t = x_0 \\ 1 & if\ t \in \mathbb{R} \setminus G_0. \end{cases}$$

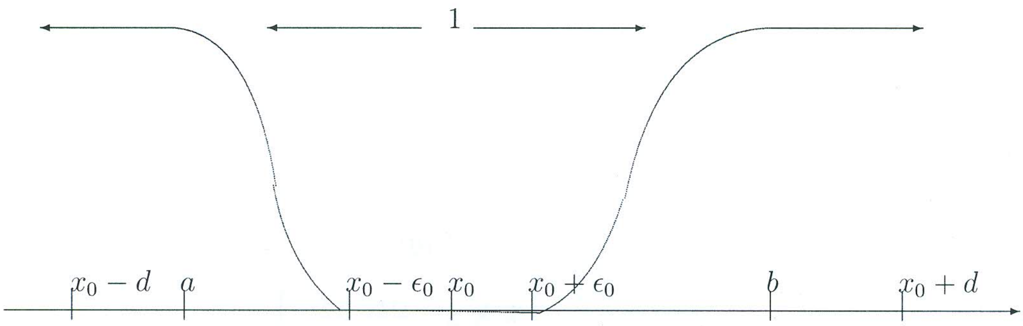

Figure 4.4: The graph of ϕ_0

$G_0 := (a, b)$

Now set $O_{x_0} := (x_0 - \epsilon_0, x_0 + \epsilon_0)$. Similarly choose open sets O_x for each $x \in f^{-1}(\lambda)$. Since $\lambda \neq 0$ and $f(x) \to 0$ as $|x| \to \infty$, $f^{-1}(\lambda)$ is a compact set and $\{O_x : x \in f^{-1}(\lambda)\}$ is an open cover for $f^{-1}(\lambda)$, hence we can find a finite sub-cover $\{O_i : i = 1, \ldots, m\} \subset \{O_x : x \in f^{-1}(\lambda)\}$. Corresponding to each O_i, let ϕ_i be the smooth function constructed above. So $\phi_i = 1$ on O_i, and $\phi_i = 1$ on $\mathbb{R} \setminus G_i$. Finally, put $\phi := \prod_{i=1}^{m} \phi_i$, $G := (\mathbb{R} \cup_{i=1}^{m} G_i)^o \supset \sigma(H)$, whence

1. ϕ is smooth on $\mathbb{R}$.

2. $\phi \equiv 0$ *on* $f^{-1}(\lambda)$.

3. $\phi \equiv 1$ *on* G.

□

Theorem 4.3.11 (Spectral Mapping Theorem)
Let $f \in \mathfrak{A}$ *and* H *is of* $(\alpha,\ \alpha+1)-$ *type* $\mathbb{R}$ *, then*

$$f(\sigma(H)) = \sigma(f(H)).$$

PROOF. Let $\lambda \in \sigma(H) \subset \mathbb{R}$ and suppose if possible

$$f(\lambda) \notin \sigma(f(H)). \tag{4.13}$$

Then $[f(\lambda) - f(H)]^{-1} \in \mathfrak{B}(\mathcal{X})$.

$$\text{If } \acute{f}_\lambda(x) := \begin{cases} \frac{f(\lambda)-f(x)}{\lambda - x}, & x \neq \lambda \\ f'(\lambda), & x = \lambda, \end{cases}$$

then by Theorem 3.3.6, $\acute{f}_\lambda \in \mathfrak{A}$ and
$(\lambda - H)\acute{f}_\lambda(H)(i-H)^{-1} = (f(\lambda) - f(H))(i-H)^{-1}$.

$$\text{Thus } (\lambda - H)\acute{f}_\lambda(H)(i-H)^{-1}(i-H)(f(\lambda)-f(H))^{-1} =$$
$$= (f(\lambda)-f(H))(i-H)^{-1}(i-H)(f(\lambda)-f(H))^{-1}$$

$\iff (\lambda - H)\acute{f}_\lambda(H)(f(\lambda)-f(H))^{-1} = I$.
Therefore $(\lambda - H)^{-1} = \acute{f}_\lambda(H)(f(\lambda)-f(H))^{-1} \in \mathfrak{B}(\mathcal{X})$!![2]

[2] We denote contradiction by !!

This contradicts the choice of λ. Hence (4.13) is not possible. Thus $f(\lambda) \in \sigma(f(H))$ implies $f(\sigma(H)) \subseteq \sigma(f(H))$.

Conversely, if $\lambda \notin f(\sigma(H))$ then $h(x) := \frac{1}{\lambda - f(x)}$ is finite for all $x \in \sigma(H)$. Moreover at each $x \in \sigma(H)$ (and $x \in G$ where G is the neighbourhood of $\sigma(H)$ constructed in lemma 4.3.10)

$$\begin{aligned}
h'(x) &= [\lambda - f(x)]^{-2} f'(x) \\
&= [h(x)]^2 f'(x) \\
h^{(2)}(x) &= f^{(2)}(x)[h(x)]^2 + 2f'(x)h(x)[h(x)]^2 f' \\
&= f^{(2)}[h(x)]^2 + 2[f'(x)]^2[h(x)]^3 \\
h^{(3)}(x) &= f^{(3)}(x)[h(x)]^2 + f^{(2)}(x)2f'(x)h(x)[h(x)]^2 f'(x) + \\
&\quad +2\{2f'(x)f^{(2)}(x)[h(x)]^3 + [f'(x)]^2 + 3[h(x)]^2[h(x)]^2 f'(x)\} \\
&= f^{(3)}(x)[h(x)]^2 + 6f'(x)f^{(2)}(x)[h(x)]^3 + 6[f'(x)]^3[h(x)]^4 \\
h^{(4)}(x) &= f^{(4)}(x)[h(x)]^2 + 8f'(x)f^{(3)}(x)[h(x)]^3 + \\
&\quad +6[f^{(2)}(x)]^2[h(x)]^3 + 36[f'(x)]^2 f^{(2)}(x)[h(x)]^4 + 24[f'(x)]^4[h(x)]^5 \\
&\vdots \quad \vdots \quad \vdots \\
h^{(m)}(x) &= \sum_{k=2}^{m+1} [h(x)]^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [f^{(i)}(x)]^{p(s,i)} l_s
\end{aligned}$$

where $l_s \in \mathbb{Z}, 1 \leq r(k) < m, 0 \leq p(s,i) \leq m$ and $\sum_{i=1}^{m} ip(s,i) = m$. Therefore since $f \in \mathfrak{A}$, we can find some $\beta < 0$ such that

$$
\begin{aligned}
|h^{(m)}(x)| &\leq \sum_{k=2}^{m+1} |h(x)|^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} |f^{(i)}(x)|^{p(s,i)} |l_s| \\
&\leq \sum_{k=2}^{m+1} |h(x)|^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [c_i \langle x \rangle^{\beta - i}]^{p(s,i)} |l_s| \\
&= \sum_{k=2}^{m+1} |h(x)|^k \sum_{s=1}^{r(k)} \langle x \rangle^{\sum_{i=1}^{m} \beta p(s,i) - \sum_{i=1}^{m} i p(s,i)} l'_s \\
&\leq \langle x \rangle^{\beta - m} \sum_{k=2}^{m+1} |h(x)|^k b_k \\
&\leq c \langle x \rangle^{\beta - m}, \; c > 0, \; \beta < 0 \qquad (4.14)
\end{aligned}
$$

(Here we have used the fact that $\sum_{i=1}^{m} p(s,i) \geq 1$ and $|h|_G < \infty$.)

If ϕ is the smooth function such that

$$
\phi(t) = \begin{cases} 0 & if \; t \in f^{-1}(\lambda) \\ 1 & if \; t \in G \end{cases}
$$

also constructed in lemma 4.3.10, set

$$
g(x) := (i - x)^{-1} \phi(x) h(x).
$$

Then using (4.14) and Lemma 3.3.5 we conclude that $g \in \mathfrak{A}$ and

$$
(\lambda - f(H)) g(H) (i - H) = I.
$$

That is, $\lambda - f(H)$ has an inverse. Therefore, $\lambda \notin \sigma(f(H))$. Hence $\sigma(f(H)) \subseteq f(\sigma(H))$. $\square$

4.4 Extending the functional calculus to $C_0(\mathbb{R})$

In this section we extend the functional calculus to $C_0(\mathbb{R})$. For more general extensions, see DeLaubenfels [DeL95]. First, we have the following preliminaries.

Let $C_0(\mathbb{R})$ denote the algebra of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(x) \to 0$ as $|x| \to \infty$ with the supremum norm $\|.\|_\infty$. Then $\mathfrak{A}$ is a dense sub-algebra of $C_0(\mathbb{R})$, Corollary 3.3.4.

Lemma 4.4.1
If $f \in \mathfrak{A}$ and H is self-adjoint on Hilbert space $\mathcal{H}$, then $\|f(H)\| \leq \|f\|_\infty$.

Proof. First, observe that H is of $(0,\ 1) - type$ $\mathbb{R}$ [proposition 2.1.1]. Also from (4.9),

$$\bar{f}(H) = f(H)^*$$

in this case. Now choose $d \in \mathbb{R}$ such that $d > \|f\|_\infty$ and set

$$g(t) := d - \sqrt{\left(d^2 - |f(t)|^2\right)}$$

then clearly $0 \leq g \in \mathfrak{A}$, and

$$(d - g(t))^2 = d^2 - |f(t)|^2, \text{ for each } t \in \mathbb{R}.$$

so

$$f\bar{f} - 2dg + g^2 = 0 \in \mathfrak{A}.$$

Thus

$$f(H)^* f(H) - dg(H) - dg(H)^* + g(H)^* g(H) = 0$$

implies $$f(H)^* f(H) + \{d - g(H)\}^* \{d - g(H)\} = d^2.$$

If $\psi \in \mathcal{H}$, then

$$\|f(H)\psi\|^2 + \|\{d - g(H)\}\psi\|^2 = d^2 \|\psi\|$$

and therefore $$\|f(H)\psi\| \leq d \|\psi\| .$$ □

We are now in a position to describe the $\mathfrak{A}$- functional calculus for a self-adjoint operator in a standard fashion.

Corollary 4.4.2

If $f \in \mathfrak{A}$ and H is self-adjoint on Hilbert space $\mathcal{H}$, then the functional calculus

$$\kappa : \mathfrak{A} \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{H})$$

can be extended to a unique map

$$\tilde{\kappa} : C_0(\mathbb{R}) \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{H})$$

such that:

1. *$\tilde{\kappa}$ is an algebra homomorphism.*
2. *$\bar{f}(H) = f(H).^*$*
3. *$\|f(H)\| \leq \|f\|_\infty$.*
4. *if $w \in \mathbb{C} \setminus \mathbb{R}$ and $r_w := (w - s)^{-1}$ then $r_w(H) = (w - H)^{-1}$.*

PROOF. The existence follows from Theorem 4.3.3, Corollary 4.3.6, Theorem 4.3.8 and Lemma 4.4.1. So we need only to establish the uniqueness.

Suppose η is another extension of κ to $C_0(\mathbb{R})$ and let $\mathfrak{X} \subseteq C_0((R)$ be the set of f for which $\tilde{\kappa}(f) = \eta(f)$. Then $\mathfrak{X}$ is norm closed sub-algebra of $C_0(\mathbb{R})$ which contains r_w for all $w \notin \mathbb{R}$. Thus whenever $x, y \in \mathbb{R}$,

$$x \neq y \iff r_w(x) \neq r_w(y) \quad \text{for some w } \notin \mathbb{R}$$

Therefore, by Stone - Weierstrass Theorem (lemma 3.3.2), $\mathfrak{X} = C_0(\mathbb{R})$. □

REMARK 4.4.3

Corollary 4.4.2 and Theorem 2.2.5 provide a proof to a version of the spectral theorem for a self-adjoint operator on a Hilbert space, which asserts:

> iH generates a uniformly bounded strongly continuous group
> if and only if H has a $C_0(\mathbb{R})$ functional calculus.

The most natural infinite dimensional analogue of a diagonalizable matrix is a **scalar operator** (short for spectral operator of scalar type in the sense of Dunford [DS71, Chapter XVIII]). For an operator H with real spectrum, this means that there exits a projection-valued measure F such that

$$Hx = \int_{\mathbb{R}} t dF(t)x$$

with maximal domain.

The class of scalar operators includes (but is not limited to) self-adjoint operators on a Hilbert space. However on a general Banach space, it is hard to find a scalar operator. If H is an operator with $\sigma(H) \subset \mathbb{R}$ and acting on a reflexive Banach space $\mathcal{X}$, then H is scalar if and only if iH generates a uniformly bounded strongly continuous group [Kan89, page

155]. So, via the spectral theorem, a self-adjoint operator H on a Hilbert space $\mathcal{H}$ is scalar if and only if H has a $C_0(\mathbb{R})$ functional calculus. In fact this is true in general. That is;

> an operator acting on a reflexive Banach space is scalar if and only if it has a $C_0(\mathbb{R})$ functional calculus [Dow78, Theorem 6.10].

In the light of the discussions in this section, it is therefore reasonable to have the following conjecture:

Conjecture 4.4.4

A densely define closed linear operator H, acting on a reflexive Banach space $\mathcal{X}$, is scalar if it is of $(0,\ 1) - type\ \mathbb{R}$ and $\|f(H)\| \leq \|f\|_\infty$ for each $f \in \mathfrak{A}$.

Notes and remarks on Chapter 4

1. For unbounded self-adjoint operators acting on a Hilbert space, Helffer and Sjöstrand [HS89] proved that (4.8) is an alternative characterisation of the standard C_0-functional calculus. Our approach has been different in that we did not assume the existence of a functional calculus but constructed one in a more general Banach space setting. In section 4.4 we showed that our functional calculus coincides with C_0-functional calculus for an unbounded operator acting on a Hilbert space.

2. W. J. Ricker [Ric88, Theorem 1], showed that the Laplacian H_0, acting on $L^p, 1 < p < \infty\ p \neq 2$ is not scaler. It is therefore the contention of conjecture 4.4.4 that $\|f(H)\| > \|f\|_\infty$ for some $f \in \mathfrak{A}$.

3. The following are known results used in this chapter

   Lemma 4.1.2 Theorem 4.1.3 Theorem 4.1.5.

4. Our contributions in this chapter include:

Lemma 4.2.2	Lemma 4.2.3	Lemma 4.2.6
Lemma 4.3.4	Lemma 4.3.10	Lemma 4.4.1
Theorem 4.2.4	Theorem 4.2.7	Theorem 4.3.1
Theorem 4.3.3	Theorem 4.3.5	Theorem 4.3.8
Theorem 4.3.11 (New proof)		
Corollary 4.3.2	Corollary 4.3.6	Corollary 4.4.2

Chapter 5

Applications of $\mathfrak{A}$ functional calculus.

In this chapter we examine properties of operators admitting $\mathfrak{A}$ functional calculus and consequences thereof.

5.1 Fractional powers of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ operators.

Theorem 5.1.1

Suppose H is of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ and $\sigma(H) \subseteq [0,\infty)$, then $H^{1/2}$ is uniquely defined and is of $(n,\ n)-type\ \mathbb{R}$ for any integer $n > \alpha+2$.

Proof. If $f(x) := (x^{1/2} - z)^{-1}$, $x > 0$, $z \notin \mathbb{R}$, then $f(H)$ is defined in the following manner:

$$g(x) := \begin{cases} \phi(x)f(x) & , 0 < x \leq 1/2 \\ f(x) & , x > 1/2 \end{cases}$$

where ϕ is the cut-off function for $[1/4, 1]$ constructed in Theorem B.0.6. Then clearly

1. $g \in C^\infty(0, \infty)$.

2. $g^{(r)}(x) \to l_r < \infty$ as $x \to 0^+$ for all $r \geq 0$.

Thus by Seeley's extension theorem (theorem 3.4.2), $Eg \in C^\infty(\mathbb{R})$. Moreover, for $x > 1/2$, we have by the Leibniz rule that

$$\begin{aligned} |g^{(r)}(x)| &= \left| \frac{d^{r-1}}{x^{r-1}} \left(-1/2 x^{-1/2} (x^{1/2} - z)^{-2}\right) \right| \\ &= \left| 2^{-r} \sum_{\nu=1}^{r} l_\nu x^{\frac{\nu-2r}{2}} (x^{1/2} - z)^{-1-\nu} \right| \\ &\quad (l_\nu \in \mathbb{Z}) \\ &\leq 2^{-r} \sum_{\nu=1}^{r} |l_\nu| \frac{\left|x^{\frac{\nu-2r}{2}}\right|}{|x^{1/2} - z|^{1+\nu}} \\ &= 2^{-r} |x|^{-r} \sum_{\nu=1}^{r} |l_\nu| \frac{|x^{\nu/2}|}{|x^{1/2} - z|^{1+\nu}}. \end{aligned}$$

Now write $z := u + iv$, whence $z \notin \mathbb{R}$ implies $v \neq 0$ and

$$|x^{1/2} - z|^2 = |(x^{1/2} - u)^2 + v^2| = \left\langle \frac{x^{1/2} - u}{v} \right\rangle^2 |v|^2. \tag{5.1}$$

In addition

$$\left\langle \frac{x^{1/2} - u}{v} \right\rangle \geq \langle z \rangle^{-1} \langle x^{1/2} \rangle, \tag{5.2}$$

by Lemma C.0.13, whence

$$\left\langle \frac{x^{1/2} - u}{v} \right\rangle^2 \geq \langle z \rangle^{-2} \langle x^{1/2} \rangle^2 \geq \langle z \rangle^{-2} \langle x \rangle \tag{5.3}$$

using $\langle \sqrt{x} \rangle^2 \geq \langle x \rangle$, see Lemma C.0.8 in Appendix C.

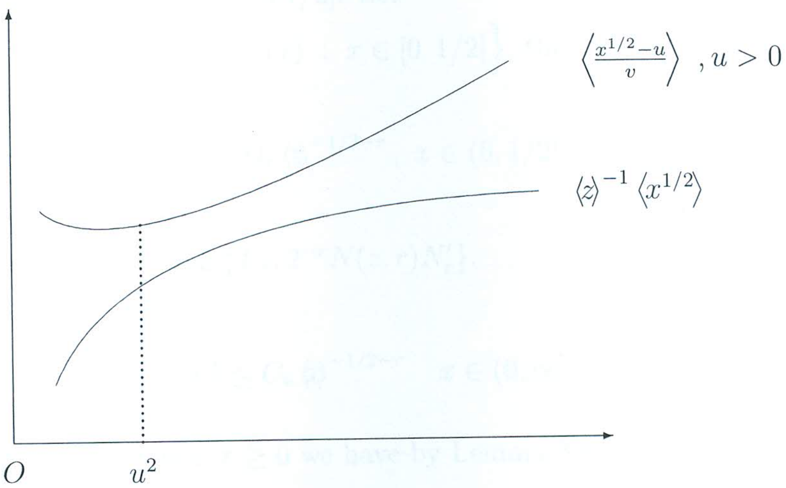

Figure 5.1: The graphs of $\left\langle \frac{x^{1/2}-u}{v} \right\rangle$ and $\langle z \rangle^{-1} \langle x^{1/2} \rangle$.

Therefore by (5.1),(5.2) and (5.3)

$$\begin{aligned} |g^{(r)}(x)| &\leq 2^{-r} |x|^{-r} \sum_{\nu=1}^{r} |l_\nu| \frac{\langle x \rangle^{\nu/2}}{\langle z \rangle^{-1-\nu} \langle x \rangle^{\frac{1+\nu}{2}}} \frac{1}{|u|^{1+\nu}} \\ &\leq 2^{-r} |x|^{-r} N(z,r) \langle x \rangle^{-1/2} \end{aligned}$$

where
$$N(z,r) = r. \max_{\nu=1,2,\ldots,r} \left\{ \frac{|l_\nu|}{\langle z \rangle^{-1-\nu} |u|^{1+\nu}} \right\}.$$

Thus
$$|g^{(r)}(x)| = 2^{-r} N(z,r) N'_r \langle x \rangle^{-1/2-r}. \tag{5.4}$$

where N'_r is a constant such that

$$\frac{1}{|x|} \leq (N'_r)^{1/r} \frac{1}{\langle x \rangle}, \quad x > 1/2, \; r > 0 \quad (\text{and } N'_r = 1 \; if \; r = 0).$$

Since $x > 1/2$ implies that $x^2 = \frac{1}{5}(4x^2 + x^2) \geq \frac{1}{5}(1 + x^2) = \frac{1}{5} \langle x \rangle^2$ we can

put $N_r' = (\frac{1}{5})^{r/2}$. Also, since $g^{(r)}$ is bounded on $(0, 1/2]$ for all r = 0,1,2,... and the function $\langle x \rangle^{r+\frac{1}{2}}$ is continuous on [0,1/2], the function $\langle x \rangle^{r+1/2} g^{(r)}(x)$ attains its bounds on [0,1/2]. Let
$D_r = \max\left\{ \langle x \rangle^{r+1/2} g^{(r)}(x) \ : \ x \in [0, 1/2] \right\}$, then

$$\left|g^{(r)}(x)\right| \leq D_r \langle x \rangle^{-1/2-r}, \ x \in (0, 1/2] \quad \text{for all } \ r \geq 0.$$

Now set $C_r := \max\{D_r, 2^{-r} N(z, r) N_r'\}$.
Then

$$\left|g^{(r)}(x)\right| \leq C_r \langle x \rangle^{-1/2-r} \quad x \in (0, \infty) \ \text{ for all } r \geq 0.$$

Therefore for each $r \geq 0$ we have by Lemma 3.4.3 that

$$\left|\frac{d^r}{dx^r}(Eg)(x)\right| \leq C_r' \langle x \rangle^{-1/2-r} \tag{5.5}$$

for some $C_r' \geq 0$, for all $r \geq 0$ and for all $x \in \mathbb{R}$ and

$$g(H) := (Eg)(H).$$

(Here, E denotes Seeley's extension operator.) Finally we now make a natural definition

$$f(H) := g(H).$$

The uniqueness follows from Theorem 4.3.5.

Next, by Theorem 4.3.5

$$\|f(H)\| \leq k \, \|f\|_{m+1}^{+}, \quad k > 0 \quad \text{whenever } m > \alpha.$$

But from (5.4)

$$\left|g^{(r)}(x)\right| = R(z,r)\,\langle x\rangle^{-1/2-r}, \quad x > 1/2$$

where

$$\begin{aligned}
R(z,r) &= 2^{-r} N_r' r \max_{\nu=1,2,\ldots,r} \left\{ \frac{|l_\nu|}{\langle z\rangle^{-1-\nu}\, |y|^{1+\nu}} \right\} \\
&= 2^{-r} N_r' r \max_{\nu=1,2,\ldots,r} \left\{ \frac{|l_\nu|\, \langle z\rangle^{1+\nu}}{|y|^{1+\nu}} \right\} \\
&\leq 2^{-r} N_r' r \max_{\nu=1,2,\ldots,r} \left\{ |l_\nu| \left(\frac{\langle z\rangle}{|y|}\right)^{\nu+1} \right\} \\
&\leq 2^{-r} N_r' r s_r \left(\frac{\langle z\rangle}{|y|}\right)^{r+1} \qquad \text{with } s_r = \max_{\nu=1,2,\ldots,r} \{|l_\nu|\}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|g\|_{m+1}^{+} &= \sum_{r=0}^{m+1} \int_0^\infty \left|g^{(r)}(x)\right| \langle x\rangle^{r-1}\, dx \\
&\leq \sum_{r=0}^{m+1} R(z,r) \left(\int_{1/2}^\infty \langle x\rangle^{-1/2-r} \langle x\rangle^{r-1}\, dx + P_r \right)
\end{aligned}$$

with

$$\begin{aligned}
P_r &:= \frac{1}{R(z,r)} \int_0^{1/2} \left|g^{(r)}(x)\right| \langle x\rangle^{r-1}\, dx \\
&\equiv \frac{T}{R(z,r)} \int_0^{1/2} \langle x\rangle^{r-1}\, dx, \quad T > 0.
\end{aligned}$$

So,

$$\|g\|_{m+1}^{+} \leq \sum_{r=1}^{m+1} R(z,r) T_r$$

$$(T_r := \int_{1/2}^{\infty} \langle x \rangle^{-1/2-1}\, dx + P_r).$$

That is

$$\begin{aligned}
\|g\|_{m+1}^{+} &\leq \sum_{r=1}^{m+1} 2^{-r} N_r' r s_r T_r \left(\frac{\langle z \rangle}{|y|}\right)^{r+1} \\
&\leq \left(\sum_{r=1}^{m+1} 2^{-r} N_r' r s_r T_r\right) \left(\frac{\langle z \rangle}{|y|}\right)^{m+2} \\
&=: C_m \left(\frac{\langle z \rangle}{|y|}\right)^{m+2}.
\end{aligned}$$

Hence by Theorem 4.3.5, we now have

$$\|f(H)\| \leq C_m \left(\frac{\langle z \rangle}{|y|}\right)^{m+2}, \qquad m > \alpha.$$

That is

$$\|f(H)\| \leq C_m \left(\frac{\langle z \rangle}{|\Im z|}\right)^{m+2}, \qquad m > \alpha$$

or

$$\|(z - H^{1/2})^{-1}\| \leq C_n \left(\frac{\langle z \rangle}{|\Im z|}\right)^{n}, \qquad n > \alpha + 2.$$

where we have put $n := m + 2$. □

5.2 Semigroups of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ operators

In this section we will have the following standing hypothesis:

The operator H is of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ acting on a Banach space $\mathcal{X}$ and H is **positive** in the sense that $\sigma(H) \subseteq [0,\infty)$.

Theorem 5.2.1

Let $f(x) := e^{-x^n t}$, $t>0$, $n \geq 1$, *then* $f(H)$ *is uniquely defined and*

$$\|f(H)\| \leq cD(n,m)F(t,m) \quad \text{some } c>0, \quad \text{whenever } m \geq \alpha \tag{5.6}$$

$$\textit{with} \quad D(n,m) = (m+2)!(\sqrt{2})^{n+m+1} \max_{1\leq r\leq m+1} \left\{ \max\{1, M'_r\} \max_{1\leq k\leq r} \{e_{r,k}(n)\} \right\},$$

$$\textit{and} \quad F(t,m) = \frac{1}{t} + \frac{1}{t}\sum_{r=1}^{m+1}\sum_{k=0}^{r-1}\frac{1}{t^k},$$

with M'_r *defined in example 3.4.5 and* $e_{r,k}(n)$ *defined in Appendix A.*

Proof. The existence and uniqueness of $f(H)$ follow from example 3.4.5 and Theorem 4.3.5.

Using (3.13) we have,

$$\begin{aligned}
\|f\|^+_{m+1} &:= \int_0^\infty \sum_{r=0}^{m+1} \left|f^{(r)}(x)\right| \langle x\rangle^{r-1}\,dx \\
&\leq \int_0^\infty \sum_{r=0}^{m+1} p_r \langle x\rangle^{-n-r} \langle x\rangle^{r-1}\,dx \\
&\quad \text{(where } p_r \text{ is defined in (3.12))} \\
&\leq \int_0^\infty \sum_{r=0}^{m+1} p_r \langle x\rangle^{-n-1}\,dx \\
&= c\sum_{r=0}^{m+1} p_r \\
&\quad \left(c := \int_0^\infty \langle x\rangle^{-n-1}\,dx\right)
\end{aligned}$$

But
$$p_r = \max\{1, M'_r\} \cdot \max_{1\leq k\leq r} \{e_{r,k}(n)\} \frac{(r+1)!\left(\sqrt{2}\right)^{n+r}}{t} \sum_{k=0}^{r-1} t^{-k}$$
$$=: \max\{1, M'_r\} \cdot q(n,r)u(t,r) \quad r \geq 1$$
and
$$p_0 = \frac{(\sqrt{2})^n}{t}$$
(Example 3.4.6)

where
$$q(n,r) := \max_{1\leq k\leq r}\left\{e_{r,k}(n)(r+1)!(\sqrt{2})^{n+r}\right\} \quad r \geq 1$$
$$= (r+1)!(\sqrt{2})^{n+r} \max_{1\leq k\leq r}\{e_{r,k}(n)\}$$
with $e_{r,k}(n)$ as in Appendix A
$$q(n,0) := (\sqrt{2})^n,$$
$$u(t,r) := \frac{1}{t}\sum_{k=0}^{r-1} t^{-k}, \quad r \geq 1 \quad \text{and}$$
$$u(t,0) := \frac{1}{t}.$$
Thus
$$\sum_{r=0}^{m+1} p_r = \sum_{r=0}^{m+1} \max\{1, M'_r\} \cdot q(n,r)u(t,r)$$
$$\leq \max_{1\leq r\leq m+1}\left\{\max\{1, M'_r\} \cdot \max_{1\leq k\leq r}\{e_{r,k}(n)\}\right\} \sum_{r=0}^{m+1}(r+1)!(\sqrt{2})^{n+r}u(t,r)$$
$$\leq \max_{1\leq r\leq m+1}\left\{\max\{1, M'_r\} \cdot \max_{1\leq k\leq r}\{e_{r,k}(n)\}\right\}(m+2)!(\sqrt{2})^{n+m+1}\sum_{r=0}^{m+1} u(t,r)$$
$$:= D(n,m)\sum_{r=0}^{m+1} u(t,r)$$
$$:= D(n,m)F(t,m)$$
which implies $\|f\|^+_{m+1} \leq cD(n,m)F(t,m).$

Thus by Theorem 3.4.4 and Theorem 4.2.7 we have

$$\|f(H)\| \leq cD(n,m)F(t.m) \quad m \geq \alpha.$$

□

Theorem 5.2.2

Let an integer $n \geq 1$ and real $t > 0$. If $f(x)$ and $g(x)$ are two smooth functions which equal $e^{-x^n t}$ for $x \geq 0$ and $f(x),\ g(x) \to 0$ as $x \to -\infty$, and H is an operator of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ for some $\alpha > 0$ then $f(H) = g(H) =: e^{-H^n t}$ and

$$e^{-H^n(t_1+t_2)} = e^{-H^n t_1} e^{-H^n t_2}$$

for all $t_1, t_2 > 0$. Moreover there exists $c_n < \infty$ such that

$$\left\|e^{-H^n t}\right\| \leq c_n \tag{5.7}$$

for all $n \geq 1$ and $0 < t \leq 1$.

Proof. $f - g = 0$ on $[0,\infty)$ and $\sigma(H) \subset [0,\infty)$. Thus from Lemma 4.3.4, we observe that $f(H) = g(H) =: e^{-H^n t}$.

Let $f_{t_1}(x) := e^{-x^n t_1}$ and $f_{t_2}(x) := e^{-x^n t_2}$.

Then $$(f_{t_1} f_{t_2})(x) := e^{-x^n t_1} e^{-x^n t_2} = e^{-x^n(t_1+t_2)}.$$

Therefore, using Theorem 4.3.3 we have

$$e^{-H^n(t_1+t_2)} = (f_1 f_2)(H) = f_1(H) f_2(H) = e^{-H^n t_1} e^{-H^n t_2}.$$

Let $f_n(x) := e^{-x^n}$, $x \geq 0$, then $f(x) = f_n(\sqrt[n]{t}x)$, $x \geq 0$, $t > 0$.

For $0 < t < 1$, set $\theta := \sqrt[n]{t}$, then $0 < \theta < 1$ and

$$\begin{aligned}\left\|e^{-H^n}\right\| &= \|f_n(H)\| \leq \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}}\tilde{f}_n(z)\right| \left\|(z-H)^{-1}\right\| dxdy \\ &\leq c\int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}}\tilde{f}_n(z)\right| \frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} dxdy \quad \text{for some } \alpha \geq 0,\ c > 0 \\ &=: cI_n.\end{aligned}$$

Also

$$\begin{aligned}\left\|e^{-H^n t}\right\| &= \|f_n(\theta H)\| \leq \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}}\tilde{f}_n(z)\right| \left\|(z-\theta H)^{-1}\right\| dxdy \\ &\leq c\int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}}\tilde{f}_n(z)\right| \frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} dxdy, \quad \text{using Theorem 2.2.8} \\ &= cI_n.\end{aligned}$$

Therefore

$$\left\|e^{-H^n t}\right\| \leq cI_n =: c_n \quad \text{for all } t \in (0,1).$$

□

REMARK 5.2.3

1. (5.7) holds for operators of $(\alpha,\ \alpha+1)'$ – *type* $\mathbb{R}$ for all $t > 0$. In this case $e^{-H^n t}$ can be defined as a bounded holomorphic semigroup for $\Re t > 0$ by a similar method, see Davies [Dav89, Theorem 2.34].

2. Since $F(t,m) = \frac{1}{t} + \frac{1}{t}\sum_{r=1}^{m+1}\sum_{k=0}^{r-1}\frac{1}{t^k}$ has a maximum at $t = 1$ on the interval $[1,\infty)$ we observe that from Theorem 5.2.1 and Theorem 5.2.2 the semigroups $e^{-H^n t}$ are uniformly bounded.

3. In the light of Theorem 5.2.2, the estimate (5.6) in Theorem 5.2.1 is much worse as $t \to 0$. In fact we prove shortly, in Theorem 5.2.6

that the semigroups $e^{-H^n t}$ are strongly continuous at $t = 0$, that is $\left\|e^{-H^n t}\right\| \to 1$ as $t \to 0$.

Corollary 5.2.4

Let A be an operator of $(\alpha,\ \alpha+1)-$type $\mathbb{R}$ with $\sigma(A) \subset [0,\infty)$, B be an operator of $(\beta,\ \beta+1)-$type $\mathbb{R}$ with $\sigma(B) \subset [0,\infty)$ and $f(x) := e^{-x^s t}$ for some real $t > 0$, some integer $s \geq 1$ and all $x \geq 0$. Then

$$\|f(A) - f(B)\| \leq c_{\alpha,\beta,s} F(t,n) \left\|(i+A)^{-1} - (i+B)^{-1}\right\|, \quad c_{\alpha,\beta,s} > 0$$

where

$$F(t,n) := \frac{1}{t} + \frac{1}{t}\sum_{r=1}^{n+1}\sum_{k=0}^{r-1}\frac{1}{t^k}, \quad t > 0$$

and $n \geq \alpha + \beta + 2$.

Proof. Since by Example 3.4.5 [(3.13)], f satisfies (3.6), $f(A)$ and $f(B)$ are defined as in the comments preceding Theorem 4.3.5, and by Theorem 5.2.2, are independent of the behaviour of f in $(-\infty, 0)$. Therefore,

$$\begin{aligned}
\|f(A) - f(B)\| &= \left\|\frac{-1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}\{(z-A)^{-1} - (z-B)^{-1}\}dxdy\right\| \\
&\leq \frac{1}{\pi}\int_{\mathbb{C}} \left|\frac{\partial \tilde{f}}{\partial \bar{z}}\right| \left\|(z-A)^{-1} - (z-B)^{-1}\right\| dxdy \\
&\leq \frac{d\left\|(i+A)^{-1} - (i+B)^{-1}\right\|}{\pi}\int_{\mathbb{C}} \left|\frac{\partial \tilde{f}}{\partial \bar{z}}\right| \frac{\langle z\rangle^{\alpha+\beta+2}}{|y|^{\alpha+\beta+2}} dxdy \\
&\quad \text{some } d > 0, \quad \text{(Proposition 2.2.14)} \\
&=: \frac{dK}{\pi}\int_{\mathbb{C}} \left|\frac{\partial \tilde{f}}{\partial \bar{z}}\right| \frac{\langle z\rangle^{\alpha+\beta+2}}{|y|^{\alpha+\beta+2}} dxdy.
\end{aligned}$$

$$\begin{aligned}
\text{Hence } \|f(A)-f(B)\| &\leq c_{\alpha,\beta}K\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}\left|f^{(r)}(x)\right|\langle x\rangle^{r}+\left|f^{(n+1)}(x)\right|\langle x\rangle^{n+1}\right)dx \quad , \\
& \quad n \geq \alpha+\beta+2 \quad \text{(see proof of Theorem 4.2.4)} \\
&\leq c_{\alpha,\beta}K\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n+1}p_r\langle x\rangle^{-s}\right)dx \quad \text{using (3.12)} \\
&=: c_{\alpha,\beta}KJ(s)\sum_{r=0}^{n+1}p_r.
\end{aligned}$$

with $J(s) := \int_{-\infty}^{\infty}\langle x\rangle^{-s}\,dx$. Therefore using the approximation for $\sum_{r=0}^{n+1}p_r$ in the proof of theorem 5.2.1,

$$\|f(A)-f(B)\| \leq c_{\alpha,\beta,s}\left\|(i+A)^{-1}-(i+B)^{-1}\right\|\left[\frac{1}{t}+\frac{1}{t}\sum_{r=1}^{n+1}\sum_{k=0}^{r-1}\frac{1}{t^k}\right]$$

where in the notations of theorem 5.2.1,

$c_{\alpha,\beta,s} := c_{\alpha,\beta}J(s)D(s,n)$. □

Remark 5.2.5

It is evident from corollary 5.2.4 that if $t > 1$ then

$$\|f(A)-f(B)\| \leq c_{\alpha,\beta,s}\left\|(i+A)^{-1}-(i+B)^{-1}\right\|\left[\frac{1}{t}+\frac{n+1}{(t-1)}\right].$$

Theorem 5.2.6

The semigroups $e^{-H^n t}$ are all strongly continuous at $t = 0$.

Proof. Since the semigroups are uniformly bounded Remark 5.2.3(2), it suffices to prove that

$$\lim_{t\to 0} e^{-H^n t}f = f$$

for all f in a dense subset of $\mathcal{X}$. We prove that

$$\lim_{t \to 0} \left\| (e^{-H^n t} - 1)(H+2)^{-1} \right\| = 0$$

for large enough m, and then let f be any element of the domain of $(H+2)$.

If $\Gamma = [0, \infty)$, $w = -2$, and $z = s$ then $w \notin \Gamma$ and hence by Lemma C.0.10 (and notations there), $\frac{1}{|-2-s|} \leq \frac{\sqrt{2}\langle -2\rangle}{\beta_0 \langle s\rangle}$ for all $s \in [0, \infty)$. Now set $g_t(s) := (e^{-s^n t} - 1)(s+2)^{-1}$, $s \geq 0$ and then proceeding as in Example 3.3.3 we have

$$\left| \frac{d^r}{ds^r}(s+2)^{-1} \right| \leq C \langle s\rangle^{-1-r} \qquad \text{for all } \ s \geq 0$$

Also, $\left| \frac{d^r}{ds^r} \left(e^{-s^n t} - 1\right) \right| \leq p_r \langle s\rangle^{-n-r}$ for all $s \in [0, \infty)$ and all $r > 0$, from (3.13).

and hence by the Leibniz formula we have

$$\left| \frac{d^r}{ds^r} [\left(e^{-s^n t} - 1\right) (s+2)^{-1}] \right| \leq D \langle s\rangle^{\beta - r}, \qquad \text{for some } \ \beta < 0 \text{ and } r > 0$$

and also

$$\begin{aligned} |g_t(s)| &= \left|(e^{-s^n t} - 1)(s+2)^{-1}\right| \\ &\leq \left|e^{-s^n t}(s+2)^{-1}\right| + \left|(s+2)^{-1}\right| \\ &\leq d_1 \langle s\rangle^{-n-1} + d_2 \langle s\rangle^{-1} \\ &\leq d_3 \langle s\rangle^{-1}. \end{aligned}$$

Hence by Lemma 3.4.3 $Eg_t \in \mathfrak{A}$.

Now,

$$\lim_{t\to 0} \|g_t\|_{m+1}^{+} = \lim_{t\,\to\,0} \int_0^\infty \sum_{r=0}^{m+1} \left|\frac{\partial^r}{\partial s^r} g_t(s)\right| \langle s\rangle^{r-1}\, ds = 0, \text{ for large enough } m.$$

Hence the result follows from Theorem 3.4.4 and Theorem 4.2.4. □

Example 5.2.7

Let $H_0 := -\Delta$ on $L^p(\mathbb{R}^N)$. Then we may write $e^{-H_0 t} f = k_t * f$, where $k_t = (2\pi N t)^{N/2} e^{-\frac{|x|^2}{2t}}$ is the Gaussian pdf with $k_t \geq 0$ and $\|k_t\|_1 = 1$. So $e^{-H_0 t}$ is actually a contraction semigroup. However $\{e^{-H_0^n t}\}_{t\geq 0}$, is a uniformly bounded semigroup, but not positivity preserving, for each $n \geq 2$.

Indeed, we can find $k_{n,t} \in L^1(\mathbb{R}^N)$ such that

$$e^{-H_0^n t} f = k_{n,t} * f \quad \text{for each } \; f \in L^p(\mathbb{R}^N)$$

See Simon [Sim82, Theorem B.7.1]. Taking Fourier transform we get

$$\hat{k}_{n,t}(x) = e^{-|x|^{2n} t}.$$

Whence by scaling we have

$$\|k_{n,t}\|_{L^1} = c_n > 1$$

for all $t > 0$. Thus on $L^1(\mathbb{R}^N)$

$$\|e^{-H_0^n t}\| = \|k_{n,t}\|_{L^1} = c_n > 1$$

So $\{e^{-H_0^n t}\}$ is uniformly bounded but not positivity preserving since otherwise $\left\|e^{-H_0^n t}\right\| \le \left\|e^{-H_0 t}\right\| \le 1$, See Davies [Dav80, Theorem 7.11]. □

REMARK 5.2.8

Note that $\left\|e^{-H^\lambda t}\right\| \le 1$ on $L^p(\mathbb{R}^N)$ for all $t \ge 0$, $0 < \lambda \le 1$, $1 \le p \le \infty$ and $N \ge 1$, because for such λ, $\{e^{-H^\lambda t}\}_{t\ge 0}$, is positivity preserving.

5.3 Operators with consistent resolvents.

Let $H_p := -\frac{1}{2}\Delta + V$ be a Schrödinger operator acting on $L^p(\mathbb{R}^n)$ with potential V from the Kato class K_n (See definition 2.1.5), and $1 \le p \le \infty$. B. Simon [Sim82] conjectured that $\sigma(H_p)$ is independent of p. This was finally proved by R. Hempel and J. Voigt [HV86] and generalised further by W. Arendt [Are94].

In this section we give a quicker proof of this result taking advantage of the functional calculus constructed here.

Recall that H_p is $(\alpha,\ \alpha+1)'-$*type* $\mathbb{R}$ with $\alpha := n\left|\frac{1}{p} - \frac{1}{2}\right|$, Theorem 2.1.7. Also Hempel and Voigt [HV86], proved that $\sigma(H_p) \subseteq [0,\infty)$.

Lemma 5.3.1

Let H be an operator admitting $\mathfrak{A}$ functional calculus. If $\lambda \in \mathbb{R}$, then $\lambda \notin \sigma(H)$ if and only if there exits $f \in C_c^\infty(\mathbb{R})$ such that $f(x) = 1$ in $(\lambda-\epsilon, \lambda+\epsilon)$ for some $\epsilon > 0$, and $f(H) = 0$.

PROOF. Suppose $\lambda \notin \sigma(H)$.

Then $(\lambda-\epsilon,\lambda+\epsilon) \cap \sigma(H) = \emptyset$ for some $\epsilon > 0$. Choose $f \in C_c^\infty(\mathbb{R})$ with $\text{supp}\,(f) \subset (\lambda-\epsilon,\lambda+\epsilon)$ using the scheme in Theorem B.0.6 in Appendix B. Then $f(H) = 0$, by Theorem 4.3.1.

Conversely, suppose there exits $f \in C_c^\infty(\mathbb{R})$ such that $f(x) = 1$ in $(\lambda - \epsilon, \lambda + \epsilon)$ for some $\epsilon > 0$, and $f(H) = 0$.

Set

$$g_w(x) \ := \ (w - x)^{-1}\{1 - f(x)\}, \quad w \notin \mathbb{R}$$

Then $g_w \in \mathfrak{A}$, by Lemma 3.3.5 and

$$\begin{aligned} g_w(H) &= (w - H)^{-1} - (w - H)^{-1} f(H) \\ &= (w - H)^{-1} \\ &= r_w(H), \quad \text{by Theorem 4.3.8,} \\ \text{where } r_w(x) &:= (w - x)^{-1} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Therefore $\quad \|g_{\lambda+i\delta}(H)\| = \|r_{\lambda+i\delta}(H)\| \leq c \, \|r_{\lambda+i\delta}\|_{n+1}$ for some $c > 0$, (4.10).

$$\begin{aligned} \text{But } \ \|r_w\|_{n+1} &= \sum_{k=0}^{n+1} \int_{-\infty}^{\infty} |r_w^{(k)}(x)| \, \langle x \rangle^{k-1} \, dx \\ &= \sum_{k=0}^{n+1} \int_{-\infty}^{\infty} c_k \, \langle x \rangle^{-k-1} \, \langle x \rangle^{k-1} \, \langle w \rangle^{r+1} \, dx \end{aligned}$$

Where we have put $c_k = \frac{n!(\sqrt{2})^{n+1}}{(\beta_0)^{(n+1)/2}}$ from (3.5), in the notations of Example 3.3.3.

$$\begin{aligned} \text{So} \qquad \|r_w\|_{n+1} &\leq \sum_{k=0}^{n+1} c_k \, \langle w \rangle^{r+1} \int_{-\infty}^{\infty} \langle x \rangle^{-1} \, dx \\ &\leq D_n \, \langle w \rangle^{n+1} \quad \text{for some } D_n > 0. \end{aligned}$$

Therefore $c \, \|r_{\lambda+i\delta}\|_{n+1} \leq cD_n \, \langle \lambda + i\delta \rangle^{n+1} \to cD_n \, \langle \lambda \rangle^{n+1}$ as $\delta \to 0$. In any case $\|f(H)\| \leq \|f\|_{n+1}$, for all $f \in \mathfrak{A}$ and some $n > \alpha$ (Theorem 4.2.4).

Thus

$$\begin{aligned}\|g_{\lambda+i\delta}(H)\| &= \|(\lambda+i\delta-H)^{-1}\| \\ &\leq c\,\|r_{\lambda+i\delta}\|_{n+1} \\ &\leq cD_n\,\langle\lambda+i\delta\rangle^{n+1} \;\rightarrow\; cD_n\,\langle\lambda\rangle^{n+1}<\infty \quad \text{as } \delta\rightarrow 0.\end{aligned}$$

Thus by Principle of Uniform Boundedness (see [DS58, Theorem II.1.18]) $(\lambda-H)^{-1}f=\lim_{\delta\to 0}(\lambda+i\delta-H)^{-1}f$ exits for each $f\in\mathcal{X}$ and is bounded. We finally invoke Lemma 4.2 of M. Schechter [Sch71], which states:

> If A is a closed operator, then $\lambda\notin\sigma(A)$ if and only if $(\lambda-A)^{-1}$ exits and is bounded on $\mathcal{X}$.

□

Definition 5.3.2

Let $\mathcal{X}$, $\mathcal{Y}$ be two Banach spaces and suppose there exist a topological vector space $\mathcal{Z}$ such that $\mathcal{X}\hookrightarrow\mathcal{Z}$ and $\mathcal{Y}\hookrightarrow\mathcal{Z}$. Two operators $H_X\in\mathfrak{B}(\mathcal{X})$ and $H_Y\in\mathfrak{B}(\mathcal{Y})$ are then said to be **consistent** if

$$H_X\phi=H_Y\phi \quad \text{for all } \phi\in\mathcal{X}\cap\mathcal{Y}.$$

Theorem 5.3.3

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces such that $\mathcal{X}\cap\mathcal{Y}$ is dense in $\mathcal{X}$ and $\mathcal{Y}$ separately. Further, let H_X act on a $\mathcal{X}$, H_Y act on $\mathcal{Y}$ and both admit 𝔄-functional calculus, and $(z-H_X)^{-1}$ and $(z-H_Y)^{-1}$ be consistent for each $z\notin\mathbb{R}$. Then $\sigma(H_X)=\sigma(H_Y)$.

Proof. If $\lambda\in\mathbb{R}\backslash\sigma(H_X)$, then there exits $f\in C_c^\infty(\mathbb{R})$ such that $f(x)=1$ in $(\lambda-\epsilon,\lambda+\epsilon)$ for some $\epsilon>0$, and $f(H_X)=0$, Lemma 5.3.1. But by consistency of the resolvents we have

$$g(H_X)\phi \quad = \quad -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{g}}{\partial\bar{z}}(z-H_X)^{-1}\phi dxdy \quad = \quad -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{g}}{\partial\bar{z}}(z-H_Y)^{-1}\phi dxdy \quad = \quad g(H_Y)\phi$$

for any $\phi \in \mathcal{X} \cap \mathcal{Y}$, and for all $g \in \mathfrak{A}$. In particular,

$$f(H_X)\phi = f(H_Y)\phi = 0 \quad \text{for all} \ \ \phi \in \mathcal{X} \cap \mathcal{Y}.$$

But $\mathcal{X} \cap \mathcal{Y}$ is dense in $\mathcal{Y}$, so $f(H_Y) = 0$ and $\lambda \notin \sigma(H_Y)$, (invoking Lemma 5.3.1 again). By symmetry we are done. □

REMARK 5.3.4

If $\{T_p(t)\}_{t\geq 0}$ is a consistent C_0-semigroup on $L^p(\Omega)$ with generator A_p ($1 \leq p < \infty$ and Ω is some open set), it is natural to ask whether the spectrum $\sigma(A_p)$ of A_p is independent of $p \in [1, \infty)$. This is not the case in general (see Hempel and Voigt [HV86] or Davies [Dav89]). In fact it is not immediate that $(z - A_p)^{-1}$ and $(z - A_q)^{-1}$ are consistent for any $z \in \rho(A_p) \cap \rho(A_q)$ (see Halberg and Taylor [HT56]). So, it is important that consistency of the resolvents be established in order to take advantage of theorem 5.3.3.

For Schrödinger operators H_p (cited at the beginning of this section), we have

Corollary 5.3.5

The spectrum of H_p is independent of p, $1 \leq p < \infty$.

PROOF. Hempel and Voigt [HV86, prop. 2.1] established the consistency of the resolvents, $(z - H_2)^{-1}$ and $(z - H_p)^{-1}$ for $1 \leq p \leq \infty$.

Secondly, $L^2 \cap L^p$ is dense in L^p for all $p \in [1, \infty)$. The conclusion now follows from Theorem 5.3.3. □

Notes and remarks on Chapter 5

1. Theorem 5.2.2 was originally formulated by Jazar [Jaz95] for a class of operators associated with spectral distributions. We have proved it here within the context of our functional calculus.

2. Our contributions in this chapter include:

   Lemma 5.3.1 Example5.2.7

   Theorem 5.1.1 Theorem 5.2.1

   Theorem 5.2.2 (new proof) Theorem 5.2.6

   Theorem 5.3.3

   Corollary 5.2.4 Corollary 5.3.5 (new proof)

Appendix A

Derivation of $\frac{d^r}{dx^r}\{e^{-x^n t}\}$

$$
\begin{aligned}
f(x) &:= e^{-x^n t} \\
\Rightarrow \quad f'(x) &= -ntx^{n-1}f(x) \\
\Rightarrow \quad f^{(2)}(x) &= -n(n-1)tx^{n-2}f(x) + (-ntx^{n-1})(-ntx^{n-1})f(x) \\
&= [-n(n-1)tx^{n-2} + (-n)^2t^2x^{2n-2}]f(x) \\
\Rightarrow \quad f^{(3)}(x) &= [-n(n-1)(n-2)tx^{n-3} + (-n)^2 2(n-1)t^2x^{2n-3}]f(x) + \\
&\quad +[(-n)^2(n-1)t^2x^{2n-3} + (-n)^3t^3x^{3n-3}]f(x) \\
&= [-n(n-1)(n-2)tx^{n-3} + 3(-n)^2(n-1)t^2x^{2n-3} + (-n)^3t^3x^{3n-3}]f(x) \\
\Rightarrow \quad f^{(4)}(x) &= [-n(n-1)(n-2)(n-3)tx^{n-4} + 3(-n)^2(n-1)(2n-3)t^2x^{2n-4} + \\
&\quad +3(-n)^3(n-1)t^3x^{3n-4}]f(x) + [(-n)^2(n-1)(n-2)t^2x^{2n-4} + \\
&\quad +3(-n)^3(n-1)t^3x^{3n-4} + (-n)^4t^4x^{4n-4})]f(x) \\
&= [-n(n-1)(n-2)(n-3)tx^{n-4} + (-n)^2(7n-11)(n-1)t^2x^{2n-4} + \\
&\quad +6(-n)^3(n-1)t^3x^{3n-4} + (-n)^4t^4x^{4n-4}]f(x) \\
\vdots \quad & \vdots \quad \vdots
\end{aligned}
$$

$$
\Rightarrow \quad f^{(r)}(x) = \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k x^{nk-r} f(x) \qquad \text{(A.1)}
$$

where $r \leq n$ and $e_{r,k}(n) \in \mathbb{Z}$.

We now establish the validity of the expansion (A.1) by induction on r, and thereby obtain a reduction formula for $e_{r,k}$.

Proposition A.0.1

Let $f(x) := e^{-x^n t}$ *for an integer* $n \geq 1$, *and all* $x \in [0, \infty)$. *Then* (A.1) *is valid for all* $0 \leq r \leq n$ *and*

$$e_{r,k}(n) = \begin{cases} \prod_{s=0}^{r-1}(n-s), & \text{if } k = 1 \\ n^r, & \text{if } k = r \\ (nk - r + 1)e_{r-1,k}(n) + ne_{r-1,k-1}(n), & \text{if } 2 \leq k \leq r-1 \end{cases}$$

PROOF. Assume (A.1) holds for some $r \in \mathbb{Z}$ and $r + 1 \leq n$. Then

$$\begin{aligned} f^{(r+1)}(x) &= \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k \frac{d}{dx}\left\{x^{nk-r} f(x)\right\} \\ &= \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k \left\{(nk-r)x^{nk-r-1} f(x) + x^{nk-r}(-n)tx^{n-1} f(x)\right\} \\ &= \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k (nk-r)x^{nk-r-1} f(x) \\ &\qquad + \sum_{k=1}^{r} e_{r,k}(n)n(-1)^{k+1} t^{k+1} x^{n(k+1)-(r+1)} f(x) \\ &= e_{r,1}(n)(-1)t(n-r)x^{n-(r+1)} f(x) \\ &\qquad + \sum_{k=2}^{r} e_{r,k}(n)(-1)^k t^k (nk-r)x^{nk-r-1} f(x) \\ &\qquad + \sum_{k=1}^{r-1} e_{r,k}(n)n(-1)^{k+1} t^{k+1} x^{n(k+1)-(r+1)} f(x) \\ &\qquad\quad + e_{r,r}(n)n(-1)^{r+1} t^{r+1} x^{n(r+1)-(r+1)} f(x). \end{aligned}$$

Now set $e_{r+1,1} := (n-r)e_{r,1}$ and $e_{r+1,r+1} := ne_{r,r}$ and adjust the index of the second summation, then

$$\begin{aligned}
f^{(r+1)}(x) &= e_{r,1}(-1)(n-r)tx^{n-(r+1)}f(x) \\
&\quad + \sum_{k=2}^{r} e_{r,k}(n)(-1)^k t^k (nk-r)x^{nk-r-1}f(x) \\
&\quad + \sum_{k=2}^{r} e_{r,k-1}(n)n(-1)^k t^k x^{nk-(r+1)}f(x) \\
&\qquad + e_{r,r}n(-1)^{r+1}t^{r+1}x^{n(r+1)-(r+1)}f(x) \\
&= e_{r,1}(-1)(n-r)tx^{n-(r+1)}f(x) \\
&\quad + \sum_{k=2}^{r} \{(nk-r)e_{r,k}(n) + ne_{r,k-1}(n)\}(-1)^k t^k x^{nk-(r+1)}f(x) \\
&\qquad + ne_{r,r}(-1)^{r+1}t^{r+1}x^{n(r+1)-(r+1)}f(x) \\
&=: \sum_{k=1}^{r+1} e_{r+1,k}(n)(-1)^k t^k x^{nk-(r+1)}f(x)
\end{aligned}$$

where

$$e_{r+1,k}(n) = \begin{cases} \prod_{s=0}^{r}(n-s), & \text{if } k=1 \\ n^{r+1}, & \text{if } k=r+1 \\ (nk-r)e_{r,k}(n) + ne_{r,k-1}(n), & \text{if } 2 \le k \le r. \end{cases}$$

The validity of the formula (A.1) for $r = 0,1,2,3,4$ has already been shown in the heuristics preceding proposition A.0.1 , hence establishing the result. □

REMARK A.0.2

For arbitrary $r \geq 0$, the expansion (A.1) is still valid except that in this case we we may define $e_{r,0} := e_{r,r+1} := 0$ for all r and then we have

$$e_{r,k}(n) = \begin{cases} 0, & \text{if } nk < r \\ (nk - r + 1)e_{r-1,k}(n) + ne_{r-1,k-1}(n), & \text{if } 1 \leq k \leq r \leq nk. \end{cases}$$

Appendix B

Cut-Off functions

Lemma B.0.3

There exists a non-negative function $\phi \in C_c^\infty(\mathbb{R})$ *with* $\phi(0) > 0$ *and* $\operatorname{supp}(\phi) \subseteq [-1, 1]$.

PROOF. Set $f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0. \end{cases}$

Then (by induction), $f^{(n)}(t) = \begin{cases} P_n(\frac{1}{t})e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0, \end{cases}$

where P_n is a polynomial of degree $2n$ and therefore $f \in C^\infty(\mathbb{R})$. Next, define ϕ by

$$\phi(x) := f(1 - |x|^2).$$

Then $\phi \in C^\infty(\mathbb{R})$ and moreover

$$|x| \geq 1 \Rightarrow \quad 1 - |x|^2 \leq 0 \Rightarrow \quad \phi(x) = 0,$$

showing that $\phi \in C_c^\infty(\mathbb{R})$. Moreover ϕ is positive if $|x| < 1$. □

REMARK B.0.4

If ϕ is the function constructed in Lemma B.0.3, then **dilation of ϕ by**

a^{-1},

$$\phi_a(x) := \phi(\frac{x}{a})$$

has similar properties as ϕ, with support in $[-a, a]$. Also **the translation of ϕ by** y,

$$\tau_y\phi(x) := \phi(x + y)$$

is such that $\tau_y\phi \in C_c^\infty(\mathbb{R})$ with support in the interval $[-y - 1, 1 - y]$.

Definition B.0.5

A function $\phi \in C_c^\infty(\mathbb{R})$ is called a **mollifier** if

1. $\phi \geq 0$ for all $x \in \mathbb{R}$
2. $\int_\infty^\infty \phi(x)dx = 1$.

Theorem B.0.6

If X is an open set in $\mathbb{R}$ and K is a compact subset of X, then one can find a mollifier, $\phi \in C_c^\infty(X)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ in a neighbourhood of K.

PROOF. Choose $\epsilon > 0$ so that $|x - y| \geq 4\epsilon$ when $x \in K$, $y \in \mathbb{R} \setminus X$. Let χ_K be the characteristic function of

$$K_{2\epsilon} := \{y \; : \; |y - x| \leq 2\epsilon \text{ for some } x \in K\}.$$

By Lemma B.0.3, we can find a non-negative function $\psi \in C_c^\infty(-1, 1)$, with $\psi(0) > 0$. Set $\psi_\epsilon(x) = \epsilon^{-1}\psi(\frac{x}{\epsilon})$, then ψ_ϵ has support in $[-\epsilon, \epsilon]$. Moreover it is easy to see that $\int \psi_\epsilon dx = 1$. Now set

$$\phi \; := \; \chi_K * \psi_\epsilon,$$

$$\begin{aligned} K_\epsilon &:= \{y : |y-x| \leq \epsilon \text{ for some } x \in K\}, \quad \text{and} \\ K_{3\epsilon} &:= \{y : |y-x| \leq 3\epsilon \text{ for some } x \in K\}. \end{aligned}$$

Then $\phi \in C_c^\infty(K_{3\epsilon})$ and since $1 * \psi_\epsilon = 1$, $1-\phi = (1-\chi_K) * \psi_\epsilon = 0$ on K_ϵ. □

Appendix C

Some useful estimates

We need the following elementary facts (stated without proof):

Lemma C.0.7

Let $z \in \mathbb{C}$, then

$$\sqrt{z} = \sqrt{1/2(|z| + \Re z)} + i\mathrm{sgn}(\Im z)\sqrt{1/2(|z| - \Re z)} \tag{C.1}$$

$$|e^z| = e^{\Re z} \tag{C.2}$$

$$|e^z| = 1 \qquad \iff z \in \mathbb{R}i \tag{C.3}$$

$$\left|\sqrt{z}\right| = \sqrt{|z|}. \tag{C.4}$$

PROOF. Standard results from the theory of Functions of Complex Variables. □

Lemma C.0.8

For any $z \in \mathbb{C}$,

$$\langle z\rangle \leq 1 + |z| \leq \sqrt{2}\,\langle z\rangle \tag{C.5}$$

$$\text{ie.} \quad \langle z\rangle \leq \langle\sqrt{z}\rangle^2 \leq \sqrt{2}\,\langle z\rangle \tag{C.6}$$

PROOF.

$$\begin{aligned} \langle z\rangle &\leq 1+|z| \\ \Longleftrightarrow \quad \langle z\rangle^2 &\leq (1+|z|)^2 \\ \Longleftrightarrow \quad 1+|z|^2 &\leq 1+|z|^2+2|z| \\ \Longleftrightarrow \quad 0 &\leq 2|z|, \quad \text{true for all } z\in\mathbb{C}. \end{aligned}$$

Also,

$$\begin{aligned} 1+|z| &\leq \sqrt{2}\,\langle z\rangle \\ \Longleftrightarrow \quad (1+|z|)^2 &\leq 2\langle z\rangle^2 \\ \Longleftrightarrow \quad 1+|z|^2+2|z| &\leq 2+2|z|^2 \\ \Longleftrightarrow \quad 2|z| &\leq 1+|z|^2 \\ \Longleftrightarrow \quad 0 &\leq 1+|z|^2-2|z| \\ \Longleftrightarrow \quad 0 &\leq (1-|z|)^2, \quad \text{true for all } z\in\mathbb{C}. \end{aligned}$$

□

Lemma C.0.9

For all $z, w \in \mathbb{C}$,

1. $\langle z \pm w\rangle \leq \sqrt{2}\,\langle z\rangle\,\langle w\rangle$
2. $\langle zw\rangle \leq \langle z\rangle\,\langle w\rangle$
3. $\left\langle \frac{z}{w}\right\rangle \leq \frac{\langle z\rangle\langle w\rangle}{|w|}$.

PROOF. We have

1.
$$\begin{aligned}
\langle z \pm w\rangle^2 &= 1 + |z \pm w|^2 \\
&\leq 1 + (|z| + |w|)^2 \\
&= 1 + |z|^2 + 2\,|z|\,|w| + |w|^2 \\
&\leq 1 + |z|^2 + |w|^2 + |z|^2\,|w|^2 + 2\,|z|\,|w| \\
&= \langle z\rangle^2 \langle w\rangle^2 + 2\,|z|\,|w| \\
&\leq \langle z\rangle^2 \langle w\rangle^2 + \langle z\rangle^2 \langle w\rangle^2 \quad (2ab \leq a^2 + b^2 < (1+a^2)(1+b^2)) \\
&= 2\,\langle z\rangle^2 \langle w\rangle^2 .
\end{aligned}$$

2.
$$\begin{aligned}
\langle zw\rangle^2 &\leq 1 + |z|^2\,|w|^2 \\
&\leq 1 + |z|^2 + |w|^2 + |z|^2\,|w|^2 \\
&= \langle z\rangle^2 \langle w\rangle^2 .
\end{aligned}$$

3. Replace w with w^{-1} in 2 to get

$$\begin{aligned}
\left\langle \frac{z}{w} \right\rangle &\leq \langle z\rangle \langle w^{-1}\rangle \\
&= \langle z\rangle \langle w\rangle\,|w|^{-1} \quad w \neq 0.
\end{aligned}$$

□

Lemma C.0.10

1. *For all* $w, z \in \mathbb{C}$,

$$\frac{1}{\langle w - z\rangle} \leq \frac{\sqrt{2}\,\langle w\rangle}{\langle z\rangle} \tag{C.7}$$

2. *Let* w *be a fixed point in* $\mathbb{C}$ *and* Γ *be any path in* $\mathbb{C}$ *with* $w \notin \Gamma$ *then,*

$$\frac{1}{|w - z|} \leq \frac{\sqrt{2}\,\langle w\rangle}{\sqrt{\beta_0}\,\langle z\rangle} \quad \textit{for all} \ \ z \in \Gamma. \tag{C.8}$$

where $\beta_0 \in (0, 1 - \langle d_0\rangle^{-1})$ *with* $d_0 := \operatorname{dist}(w,\ \Gamma)$ *and* $\operatorname{dist}(,\ )$ *is defined in* (2.10).

PROOF. Here,

1.
$$\begin{aligned} \langle z\rangle = \langle z - w + w\rangle &\leq \sqrt{2}\,\langle w - z\rangle\,\langle w\rangle \quad \text{(lemma C.0.9)} \\ \Longleftrightarrow \frac{1}{\langle w - z\rangle} &\leq \frac{\sqrt{2}\,\langle w\rangle}{\langle z\rangle}. \end{aligned}$$

2.
$$\begin{aligned} \frac{1}{|w - z|^2} &= \frac{1}{\langle w - z\rangle^2 - 1} \\ &= \frac{1}{(\langle w - z\rangle - 1)(\langle w - z\rangle + 1)} \qquad \text{(C.9)} \\ &\leq \frac{1}{(\langle w - z\rangle - 1)\,\langle w - z\rangle}. \qquad \text{(C.10)} \end{aligned}$$

Since $w \notin \Gamma$, $\langle w - z\rangle - 1 > 0$ for all $z \in \Gamma$, that is $0 < \langle w - z\rangle - 1 < \langle w - z\rangle$ for all $z \in \Gamma$, we observe that for a given $z \in \Gamma$ we can find $\alpha_z \in \mathbb{R}$ such that

$$0 < \alpha_z < 1 \text{ and } \langle w - z\rangle - 1 = \alpha_z\,\langle w - z\rangle\,.$$

Moreover

$$\beta \langle w - z \rangle < \langle w - z \rangle - 1 \quad \text{for all } \beta \text{ such that } 0 < \beta < \alpha_z.$$

In fact the map

$$\begin{aligned} \phi : \Gamma &\to \mathbb{R} \\ z &\mapsto 1 - \langle w - z \rangle^{-1} \end{aligned}$$

is continuous, bounded above by 1 and below by 0 and has only one minimum,

$\alpha := \phi(z_w) = 1 - \langle d_0 \rangle^{-1} > 0$ (with $d_0 = \text{dist}\,(w,\ \Gamma) > 0$).

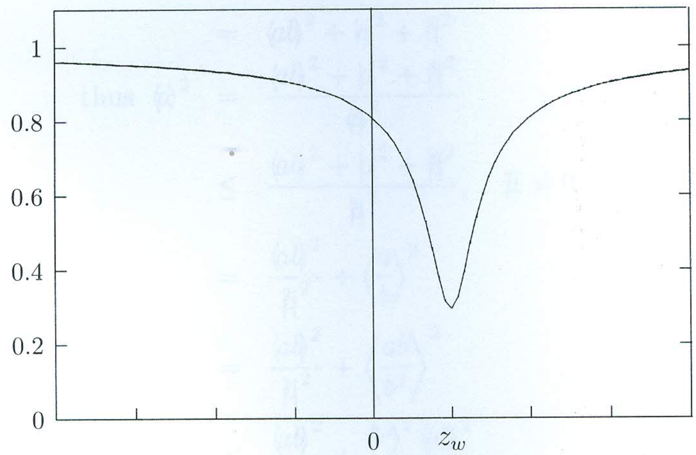

Figure C.1: A 2D view parallel to the $\mathbb{C}$-plane, of the graph of ϕ

Next, choose $\beta_0 \in (0, \alpha)$, small enough so that

$$\beta_0 \langle w - z \rangle < \langle w - z \rangle - 1 \quad \text{for all } z \in \Gamma.$$

Then

$$\frac{1}{|w-z|^2} \leq \frac{1}{\beta_0 \langle w-z\rangle^2} \quad \text{for all } \; z \in \Gamma.$$

Thus (C.8) now follows from (C.7).

□

Lemma C.0.11

$$\langle a\rangle \leq \frac{\langle ab\rangle}{|b|}\left\langle \frac{\langle b^2\rangle}{|b|}\right\rangle, \quad b \neq 0.$$

PROOF.

$$\begin{aligned}
\langle a\rangle^2 \langle b\rangle^2 &= 1+|a|^2+|b|^2+|a|^2|b|^2 \\
&= \langle ab\rangle^2+|a|^2+|b|^2 \\
\text{thus } \langle a\rangle^2 &= \frac{\langle ab\rangle^2+|a|^2+|b|^2}{\langle b\rangle^2} \\
&\leq \frac{\langle ab\rangle^2+|a|^2+|b|^2}{|b|^2}, \quad |b| \neq 0 \\
&= \frac{\langle ab\rangle^2}{|b|^2}+\left\langle \frac{a}{b}\right\rangle^2 \\
&= \frac{\langle ab\rangle^2}{|b|^2}+\left\langle \frac{ab}{b^2}\right\rangle^2 \\
&\leq \frac{\langle ab\rangle^2}{|b|^2}+\frac{\langle b^2\rangle^2\langle ab\rangle^2}{|b|^4} \\
&\quad \left(\text{using } \left\langle \frac{m}{n}\right\rangle^2 \leq \frac{\langle m\rangle^2\langle n\rangle^2}{|n|^2}, \quad \text{Lemma } C.0.9\right) \\
&= \frac{\langle ab\rangle^2}{|b|^2}\left(1+\frac{\langle b^2\rangle^2}{|b|^2}\right) \\
&= \frac{\langle ab\rangle^2}{|b|^2}\left\langle \frac{\langle b^2\rangle}{|b|}\right\rangle^2.
\end{aligned}$$

□

Lemma C.0.12

For any $x + iy \;:=\; z \in \mathbb{C}$,

$$\frac{\partial}{\partial x} \langle z \rangle^{\lambda} \leq \lambda \langle z \rangle^{\lambda - 1} \quad \text{and} \quad \frac{\partial}{\partial y} \langle z \rangle^{\lambda} \leq \lambda \langle z \rangle^{\lambda - 1}$$

PROOF.

$$\begin{aligned} \frac{\partial}{\partial x} \langle z \rangle^{\lambda} &= \frac{\partial}{\partial x} (1 + |x|^2 + |y|^2)^{\lambda/2} \\ &= \frac{\lambda}{2} (1 + |x|^2 + |y|^2)^{\lambda/2 - 1} \cdot 2\,|x| \\ &= \lambda\,|x|\,\langle z \rangle^{\lambda - 2} \\ &\leq \lambda \langle z \rangle^{\lambda - 1}. \end{aligned}$$

The proof for $\frac{\partial}{\partial y} \langle z \rangle^{\lambda} \leq \lambda \langle z \rangle^{\lambda - 1}$ is analogous to the one above. □

Lemma C.0.13

Let $z := u + iv$, $u, v \in \mathbb{R}$ *with* $v \neq 0$. *Then* $\left\langle \frac{|x|^{1/2} - u}{v} \right\rangle \geq \langle z \rangle^{-1} \left\langle |x|^{1/2} \right\rangle$ *for all* $x \in \mathbb{R}$.

PROOF.

$$\begin{aligned} \frac{\left\langle |x|^{1/2} \right\rangle^2}{\left\langle \frac{|x|^{1/2} - u}{v} \right\rangle^2} &\leq \langle z \rangle^2 \qquad \text{(C.11)} \\ \Longleftrightarrow\; 1 + |x| &\leq (1 + u^2 + v^2) \left(1 + \frac{(|x|^{1/2} - u)^2}{v^2} \right) \end{aligned}$$

put $s := |x|^{1/2} - u$, then $x = (u + s)^2$.

Therefore, using (C.11)

$$
\begin{array}{rrcl}
 & 1+u^2+2us+s^2 & \leq & (1+u^2+v^2)(1+\frac{s^2}{v^2}) \\
 & & = & 1+u^2+v^2+\frac{s^2}{v^2}+\frac{u^2s^2}{v^2}+s^2 \\
\iff & 2us & \leq & v^2+\frac{s^2}{v^2}+\frac{u^2s^2}{v^2} \\
\iff & 0 & \leq & \left(v^2-2\frac{us}{v}v+\frac{u^2s^2}{v^2}\right)+\frac{s^2}{v^2} \\
 & & = & \left(v-\frac{us}{v}\right)^2+\frac{s^2}{v^2}. \\
 & & & \text{True for all } v,s,u\in\mathbb{R}.
\end{array}
$$

□

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