THE DERIVATION OF A LOGISTIC NONLINEAR BLACK SCHOLES MERTON PARTIAL DIFFERENTIAL EQUATION:EUROPEAN OPTION

by

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A thesis submitted in fulfilment
of the requirements for the award of the degree of
Doctor of Philosophy in Applied Mathematics

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Abstract

Nonlinear Black-Scholes equations have been increasingly attracting interest over the last twenty years. This is because they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, illiquid markets, risks from an unprotected portfolio or large investor's preferences, which may have an impact on the stock price, the volatility, the drift and the option price itself. Recent models have been developed to take into account the feedback effect of a fund hedging strategy or of the transaction costs of large traders. Most of these models are represented by nonlinear variations of the well known Black-Scholes Equation.On the other hand, asset security prices may naturally not shoot up indefinitely (exponentially) leading to the use of Verhlust's Logistic equation. The objective of this study was to derive a Logistic Nonlinear Black Scholes Merton Partial Differential equation by considering transaction costs (which were overlooked in the derivation of the classical Black Scholes model) and incorporating the Logistic geometric Brownian motion.The methodology involved, analysis of the geometric Brownian motion, review of logistic models, Itô's process and lemma, stochastic volatility models and the derivation of the linear and nonlinear Black-Scholes-Merton partial differential equation .Illiquid markets have also been analyzed alongside stochastic differential equations. The result of this study may enhance reliable decision making based on a rational prediction of the future asset prices given that in reality the stock market may depict a non linear pattern.

Chapter 1

INTRODUCTION

1.1 Background information

Stochastic differential equations are fundamental in describing and understanding random phenomena in different areas in physics, engineering, finance, economics and other areas. In particular, they serve as a model for asset price fluctuation in finance and is the driving force behind the famous Black-Scholes-Merton option pricing partial differential equation used for deriving the Black-Scholes-Merton model in the year 1973

Whereas ordinary differential equations are usually interpreted as describing evolution in time and hence determining dynamic systems, the future becomes quite predictable from the knowledge of the present state. There are however systems which are known to exhibit randomness and hence are non-deterministic, like the predator-prey ecosystem and the markets. This leads to partial differential equations(which involve more than one independent variable and their corresponding derivatives).

In the derivation of the Logistic Nonlinear Black-Scholes-Merton partial differential equation, three processes shall be considered namely:

(1)The arithmetic Brownian motion(Bachelier process)

$$dS = \mu dt + \sigma dZ \tag{1.1}$$

(2)The Itô process

$$dS(t) = f(S,t)dt + f(S,t)dZ \tag{1.2}$$

(3)The geometric Brownian motion(Wiener process), which is a special type of the Itô process in which $f(S,t) = S$

$$dS = \mu S dt + \sigma S dZ \tag{1.3}$$

where S is the stock price, μ is the expected rate of return per unit time and σ is the volatility of the stock price, whereas dZ is the addition of noise.

Of necessity, we note that in stock price modeling the price of an asset is assumed to respond to the excess demand which is the difference between the quantity of an asset demanded and the quantity of the same asset supplied, That is,

$$EDS(t) = Q_D S(t) - Q_S s(t) \tag{1.4}$$

where:

$EDS(t)$ is the excess demand,$Q_D S(t)$ and $Q_S S(t)$ are the quantities demanded and supplied respectively at a given time,t and price, S(t)

Just like in the predator-prey ecosystem where there is "give and take", the market structure with forces of supply and demand exhibit two forces in the market which affect each other striving to strike a balance called the market equilibrium.

This comparative phenomenon has made it possible to apply the idea of logistic equation, first used by Verhlust,and Reed.([36],[37],[38])

In Verhlust's model for studying dynamics of human population growth in the United states, he took p^* to represent the environmental carrying capacity in which a population lives, which favorably compares to s^* in the Walrasian equilibrium market price, a point where the quantity supplied and demanded in the market are equal.
In this study we intend to formulate the Logistic Nonlinear Black-Scholes-Merton partial differential equation.

1.2 Statement of the problem

Although much work has been done to model the nonlinear Black-Scholes-Merton partial differential equation, The Logistic Nonlinear Black-Scholes-Merton partial differential equation has not been derived. In this study we consider the Logistic Geometric Brownian motion alongside more realistic assumptions such as transaction costs, risk from unprotected portfolio, large investor's preferences or illiquid markets, which may have an impact on the stock price, the volatility, the drift and the stock price itself.

1.3 Objective of the study

The objective of this study was to derive a Logistic Nonlinear Black Scholes Merton Partial Differential equation by considering transaction costs (which were overlooked in the derivation of the classical Black Scholes model) and incorporating the Logistic Geometric Brownian motion which was not used in the derivation of the recent linear models.

1.4 Significance of the study

The Solution to the Partial Differential Equation derived in this study may enhance reliable decision making based on a rational prediction of the future asset prices given that in reality the stock market may depict a non linear pattern.

1.5 Research Methodology

The methodology involved, analysis of the geometric Brownian motion, review of logistic models, Itô's process and lemma and stochastic volatility models. Illiquid markets have also been analyzed. An analysis of stochastic differential equations has also been done extensively.

Chapter 2

LITERATURE REVIEW

2.1 Linear models

Brown in the year 1827. first observed the continuous movement of pollen particles suspended in a liquid. In 1905 Einstein derived the mathematics of random walk from the conservation equation with an empirical law of physics and eventually obtained the diffusion coefficient in physics known as diffusivity, which later came to be referred to in price dynamics as volatility. Louis Bachalier a contemporary of Einstein applied this random phenomenon(stochastic process) in his PhD thesis titled *Théorié de la speculation*(The theory of speculation) ([2]).

Wiener introduced rigorous mathematical and probabilistic concepts and proof that resulted into the theory of stochastic process(also known as Weiner process). Later Itô gave a rigorous treatment to stochastic process and stochastic differential equations and ended up with the laws that govern stochastic integration and solutions to stochastic differential equations, hence the norm of Itô processes and Itô's lemma, also Itô's laws.

However it was not until 1960s and 1970s when applications of stochastic processes started finding inroads into the financial markets. Samuelson in the year 1965 developed a geometric Brownian Motion (also known as the economic or exponential model) that became an alternative to Bachalier's stochastic model. The disadvantage of Bachalier's stochastic model was that it allowed negative asset prices([18],[45]).

Black and Scholes together with Merton in 1973 made a major breakthrough in pricing stock options in the celebrated Black-Scholes-Merton option pricing model(BSMOPM). They used geometric Brownian motion to derive the formula, which has become a benchmark in option pricing and has been researched by both practitioners and in academia ([7],[33]). Many researchers have modified and relaxed assumptions due to the Black-Scholes-Merton model thus several models have been developed, amongst others:

Hull and White[1987]([16]) developed a stochastic volatility model due to the fact that in real sense volatility may not always be constant. In the year 2003, Onyango ([37]) using Walrasian excess demand principle developed a logistic equation for asset security prices considering the fact that naturally asset prices would not usually shoot indefinitely(exponentially) due to a regulating factor that may limit the asset prices. Recent work has focussed on steady market conditions defined by the Itô process.

$$dS = \mu S dt + \sigma S dZ, \tag{2.1}$$

where the drift coefficient μ reflects price trading driven by constant investors' expectation of gain, while the diffusive Wiener process σdZ reflects the response of trading random fluctuations of supply and demand.This leads to the Itô process of the form

$$dC = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S}\sigma S dZ, \tag{2.2}$$

Hence the corresponding Black-Scholes-Merton partial differential equation to (2.1)is given by

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \tag{2.3}$$

where r is a risk free interest rate , C is the price of the call option while S is the spot price of an asset at time zero.

Other developments that have come up are by Cox and Ross in 1976 on stochastic volatility model, and borrowing the mean reversion model by Ornstein and Uhlensbeck in 1930 from physics ([9],[39]).

Given that historical volatility estimates in a moving window may be *heteroskedastic* (stock volatility varies over time with periods of high volatilities and periods of calm) and not always *homeskedastic* (constant volatility). Onyango considering excess demand developed an Itô process of the form

$$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ \tag{2.4}$$

where S^* is the equilibrium market price of an asset, S is the market price of the asset and μ is the speed of market adjustment, σ is constant volatility and dZ is the Wiener process ([37]).

2.2 Nonlinear models

The nonlinear dynamic process was first noticed by Mandelbrot in the year 1963, he established that speculative market prices followed a fractional distribution leading to the fractional Brownian motion (FBM). Years later in 1999 Lo and Mackinlay came to the same conclusion as a followup to Muller's detection (in 1990) of non linear market dynamics. These findings were later confirmed by Karuppiah and Los in the year 2005 ([30],[27],[34],[22]).

Nonlinear dynamic systems became of interest since it offered a new approach to financial predictability as observed by Franses and Van Dijk in the year 2000. The models therefore became useful in predicting asset prices in the long run while linear models only proved useful in the short run as observed by Savit(in the year 1988). This is because Linear models are modeled under a number of assumptions which in most cases may not reflect the reality in the market ([52],[47]).

Works by Jarrow in 1994,Platen and Schweizerin 1998 ,Frey in 1998 , Frey and Streme in 2000, Sicar and Papanicolaou in 1998, Wilmott in the year 2000 and Baum in the year 2001 among others have contributed richly to Nonlinear models of finance. In all the cases however, the geometric Brownian motion was used ([19],[41],[12],[14],[51],[58],[5]).

In Muhannad's analysis on stock prices for the European option in the year 2000 for the Log-normal Model and the Log-logistic models, he proved that option prices were overpriced by the Log-Normal model as opposed to the Log-logistic model . This gives room for us to analyze the effect of using the logistic Geometric Brownian motion ([35]).

Chapter 3

BASIC CONCEPTS

3.1 Stock Market

A stock market is a designated place where stock traders transact shares and securities. In a stock market, stocks are floated for purchase and sale. The following definitions are therefore necessary in our study as far as stock markets are concerned.

3.1.1 Stock prices and strike prices

The stock price is the price of an asset at a given time t, while the strike price (exercise price) is the specified asset buying or selling price in future.

3.1.2 Options

An option gives its owner the right but not the obligation to buy (in case of call option) or sell (in case of put options) a certain quantity of an asset by a certain future date at an agreed price.

3.1.3 Types of options

There are two main types of options which we define as:

Call option-This type of option gives the holder the right to buy an underlying asset by a certain date at a certain price.It therefore gives one the option of calling for stock at a specified price(strike price).

Put option -This type of option gives the holder the right to sell the underlying asset by a certain date for a certain price

The buyer or seller in both cases is not under compulsion to exercise the option but may chose to exercise the option or fail to do so depending on market prospects .

3.1.4 Styles of Option Contracts

The two most popular styles of options are:

i)**European style option**- This kind of contract can only be exercised at maturity date(on the date of maturity).

ii)**American style option**-This type of contract can be exercised any time prior to the maturity date.

In this study we shall concentrate on the European style option.

3.1.5 Portfolio

This is a list of security holdings by an individual, a bank or an investment company or any given investor. Since in every investment profit is expected, The choice of a portfolio is crucial to the portfolio holder in a stock market.

3.1.6 Derivatives

These are financial instruments in the stock market whose prices depend on, or is derived from the price of other underlings assets or primary financial quantities such as stocks interest rates, currencies and commodities [17].

3.1.7 Hedgers

Hedging involves all activities that are aimed at reducing the unpredictability of certain unknown future prices. Hedgers are therefore interested in reducing the risk they already have by making sure that they use the market to take cover against unpleasant asset price movements.

3.1.8 Delta Hedging

This is the perfect elimination of risks by employing correlation between two instruments (in this case an option and its underlying)([56],[57],[58])

3.1.9 Arbitrageurs

These are people or portfolio holders whose main interest is to enter into two or more markets simultaneously in order to make risk free profit. A non-arbitrage principle is therefore in place suggesting that it is unnatural to make profit with zero investment and without risk bearing.

3.2 Stochastic Processes

3.2.1 Definition of stochastic processes

A variable whose value changes randomly (in an uncertain way)is said to follow a stochastic process. It could also be defined as a sequence of events governed by chance (probabilistic laws). This is a process that can occupy one among a number of states at any given time and which with certain probability, makes transition from one state to another as time progresses. The set of possible states may be finite or infinite ([17],[37],[57]).

3.2.2 The Markov process

In this type of stochastic process only the current value of a variable is relevant for predicting the future. The past history or the way the present has emerged is not important (the past is irrelevant) since it is believed that the current price already contain what is relevant from the past. Stock prices are assumed to follow the Markov process.

By this Markov property it is implied that the probability distribution of the price at any particular future time is not dependant on a particular path followed in the past. Thus the past does not determine the future. In a Markov process the variance of change in successive time is additive while the standard deviation is not. ([17],[37]),

3.2.3 Wiener process/Brownian motion

This is a type of a markov process with a mean change of zero and a variance of 1 in a given period of time. If $X(t)$ follows a stochastic process where μ is the mean and σ is the standard deviation,
that is $X(t) \sim N(\mu, \sigma)$ then for a standard Wiener process, $X(t) \sim N(0, 1)$, that is $X(t)$ is a normal distribution whith $\mu = 0$ and $\sigma = 1$.

Expressed formally, a variable Z follows a Wiener process if it has two properties:-

3.2.3.1 The change in Z, δZ during a small period of time δt is

$$\delta Z = \epsilon\sqrt{\delta t}, \tag{3.1}$$

where ϵ is a random drawing from a standardized normal distribution , that is $\epsilon \sim N(0, 1)$

3.2.3.2 The value of δZ for any two different short intervals of time δt are independent. That is $var(\delta Z_i, \delta Z_j) = 0, i \neq j$

It follows from property (3.2.3.1) that δZ itself has a distribution with

$$\text{mean of } \delta Z = 0$$
$$\text{standard deviation } \delta Z = \sqrt{\delta t}$$
$$\text{Variance of } \delta Z = \delta t.$$

that is,

$$\delta Z \sim N(0, \sqrt{\delta t})$$

Property (3.2.3.2) implies that t follows a Markov process. Consider the increase in the value of Z during a relatively long period of time T, this can be denoted by $Z(T) - Z(0)$. It can be regarded as the sum of increase in Z in n small intervals where $n = \frac{T}{\delta t}$ thus

$$Z(T) - Z(0) = \sum_{i=1}^{n} \epsilon_i \sqrt{\delta t}, \tag{3.2}$$

where the $\epsilon_i (i = 1, 2, 3....n)$ are random drawings from $N(0, 1)$. From property (3.2.3.2) of the Wiener process, the ϵ_is are independent and identically distributed. It follows from equation (3.2) that $Z(T) - Z(0)$ is normally distributed with

$$\text{mean} = E\Big(Z(T) - Z(0)\Big) = 0$$
$$\text{variance of } \Big(Z(T) - Z(0)\Big) = n\delta t = T, \text{ and,}$$
$$\text{standard deviation of } \Big(Z(T) - Z(0)\Big) \text{ is } \sqrt{T},$$

hence,

$$Z(T) - Z(0) \sim N(0, \sqrt{T})$$

. ([17],[37]).

3.2.4 The Generalized Wiener process

So far the standard Wiener process dZ, has a drift rate of zero and variance of 1. This implies that the expected value of Z at any future time is equal to its current value whereas the variance rate 1, means that the variance of change in Z in time interval of length T equals T

A generalized Wiener process for a variable X can be defined in terms of dZ as follows

$$dX = adt + bdZ, \tag{3.3}$$

for

$$X(0) = X_0, t \geq 0,$$

where a and b are constants, adt is the expectation of dX and bdz is the addition of noise or variability to the path followed by X while b is the diffusivity. In a small interval δt, the change in the value of X, δX is of the form $\delta X = a\delta t + b\epsilon\sqrt{\delta t}$ where as already defined, ϵ is a random variable drawing from a standardized normal distribution thus δX has a normal distribution with

$$\text{mean} = \text{E}(\delta X) = a\delta t$$
$$\text{variance of } (\delta X) = b^2\delta t, \text{ and,}$$
$$\text{standard deviation of } \delta X = b\sqrt{\delta t},$$

hence

$$\delta X \sim N(a\delta t, b\sqrt{\delta t})$$

Similar arguments to this show that for a Wiener process, the changes in the value X in a time interval T is normally distributed with,

$$\text{mean change in } X = aT$$
$$\text{standard deviation of change in } X = b\sqrt{T}$$
$$\text{variance of change in } X = b^2T.,$$

that is,

$$dX \sim N(aT, b\sqrt{T})$$

([17],[36],[37],[38],[57],[58])

3.2.5 Itô Process

This is a generalized Wiener process where the parameter a and b are functions of the value of the underlying variables X and time t .
Algebraically the Itô process can be written as

$$dX = a(X,t)dt + b(X,t)dZ \tag{3.4}$$

([17],[37])
This implies that both the expected drift rate and variance of an Itô process can undergo change.
In a small time interval between t and $t + \delta t$ the variable changes from X to $X + \delta X$ where;

$$\delta X = a(X,t)\delta t + b(X,t)\epsilon\sqrt{\delta t}. \tag{3.5}$$

([17],[37])
This relationship involves a small approximation which assumes that the drift and variance rate of X remains constant; equal to $a(X,t)$ and $b^2(X,t)^2$ respectively during the interval between t and $t + \delta t$
that is,

$$dX \sim N\left(a(X,t), b(X,t)\sqrt{\delta t}\right)$$

3.2.6 Geometric Brownian Motion

A specific type of Itô process is the geometric Brownian motion of the form

$$dX = aXdt + bXdZ \text{ where } a(X,t) = aX \text{ and } b(X,t) = bX \qquad (3.6)$$

The geometric Brownian motion has been applied in stock pricing and is given as equation (1.3) which can also be written as

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad (3.7)$$

This model (3.7) is the most widely used model of stock price behavior. A review of this model gives a discrete time model

$$\frac{\delta S}{S} = \mu\delta t + \sigma\epsilon\sqrt{\delta t}, \qquad (3.8)$$

where δS is the change in stock price S in a small interval of time δt and ϵ is a random variable drawn from a standardized normal distribution $N(0,1)$. Hence in a short time δt, the expected value of return is $\mu\delta t$ and the stochastic component of the return is $\sigma\epsilon\sqrt{\delta t}$. The variance of the fractional rate of return is $\sigma^2\delta t$ and $\sigma\sqrt{\delta t}$ is the standard deviation. Therefore $\frac{\delta S}{S}$ is normally distributed with mean $\mu\delta t$ and standard deviation, $\sigma\sqrt{\delta t}$, that is

$$\frac{\delta S}{S} \sim N(\mu\delta t, \sigma\sqrt{\delta t})$$

3.2.7 Volatility

Volatility is the measure of how uncertain we are about future stock price movement. The volatility of a stock price σ is defined so that $\sigma\sqrt{\delta t}$ is the standard deviation of the return on stock in a short period of time δt. As volatility increases therefore ,the chance that a stock will do very well or very poorly increases. This results in both the call and put options rising or falling respectively.

3.2.8 Stochastic Volatility

One assumption in the Black-Scholes-Merton model is that volatility is always constant. However Hull and White [1987] among others considered stochastic volatility models. They considered the fact that in a real market situation volatility may follow a stochastic process of the form

$$d\sigma = \mu_\sigma \sigma dt + v_\sigma \sigma dZ \tag{3.9}$$

or

$$d\sigma = \mu_\sigma (b - \sigma) dt + v_\sigma \sigma dZ, \tag{3.10}$$

where μ, b and v are constants and dZ refers to the Wiener process, σ is the asset volatility while μ_σ and v_σ the mean and variance of asset volatility respectively. In equation (3.10) the variance rate has a drift that pulls it back to a level b at a rate μ_σ.

3.3 Itô's lemma and its derivation

Itò, achieved a rigorous treatment for integrating a wide range of Wiener-like differential processes into a strict mathematical framework. Itô's lemma is therefore used to solve stochastic differential equations which is analogous to the chain rule in Newtonian calculus.[16],[17]

Suppose that the value of S follows an Itô process

$$dS = \mu(S,t)dt + \sigma(S,t)dZ$$

where dZ is the Wiener process and μ and σ are functions of S and t and the variable S has a drift rate of $\mu(S,t)$ and variance rate of $\sigma(S,t)$. Itô's lemma states that a continuous function $G(S,t) = G$, that is twice differentiable in S and once in t, follows an Itô process given by ([17],[56],[57])

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dZ \tag{3.11}$$

Consider a continuous function $G(X)$ that is twice differentiable in X. If δX is a small change in X and δG the corresponding change in G then

$$\delta G \simeq \frac{dG}{dX}\delta X \tag{3.12}$$

δG being approximately equal to the product of the rate of change with respect to X and δX. For more accuracy , the Taylor series of δG is given as,

$$\delta G = \frac{dG}{dX}\delta X + \frac{1}{2}\frac{d^2G}{dX^2}\delta X^2 + \frac{1}{6}\frac{d^3G}{dX^3}\delta X^3 + \ldots$$

in which the terms in δX^2 and above are considered to be too small in equation (2.3). For G continuous and differentiable in two variables X and Y, the result analogous to equation (3.12) would be

$$\delta G \simeq \frac{\partial G}{\partial X}\delta X + \frac{\partial G}{\partial Y}\delta Y \tag{3.13}$$

whose second order Taylor series expansion is

$$\delta G = \frac{\partial G}{\partial X}\delta X + \frac{\partial G}{\partial Y}\delta Y + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\delta X^2 + \frac{\partial^2 G}{\partial X \partial Y}\delta X \delta Y + \frac{1}{2}\frac{\partial^2 G}{\partial Y^2}\delta Y^2 + \ldots \tag{3.14}$$

Taking limits as δX and δY tend to zero, equation (3.14) becomes

$$dG = \frac{\partial G}{\partial X}dX + \frac{\partial G}{\partial Y}dY \tag{3.15}$$

Suppose the variable X follows an Itô's process (equation 3.4) and that G is a function of X and of time t then we can from equation (3.14) extend equation (3.15) to cover functions that follow Itô processes hence by analogy, equation (3.14) becomes

$$\delta G = \frac{\partial G}{\partial X}\delta X + \frac{\partial G}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\delta X^2 + \frac{\partial^2 G}{\partial X \partial t}\delta X \delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\delta t^2 + \ldots \tag{3.16}$$

Equation (3.4) can be discretized to form equation (3.5) which can also be written as

$$\delta X = a\delta t + b\epsilon\sqrt{\delta t} \tag{3.17}$$

We observe here that there is a significant difference between equation (3.16) and equation (3.14), because although all the terms in δX^2 are dropped in equation (3.15) (since they are too small), equation (3.17) however indicates clearly that

$$\delta X^2 = b^2\epsilon^2\delta t + \text{terms of higher order in } \delta t. \tag{3.18}$$

hence terms in δX^2 cannot be ignored since it contains δt as a component. From equation (3.18) therefore conclude that δX^2 becomes nonstochastic (deterministic) and is equal to b^2dt as δt tends to zero as can be seen from the multiplication table below,

X	dZ	dt
dZ	dt	0
dt	0	0

Taking limits as δX and δt tends to zero in equation (3.16) and ignoring the terms in δX^3 and δt^2 and higher terms we obtain

$$dG = \frac{\partial G}{\partial X}dX + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2dt \tag{3.19}$$

which is the Itô's lemma. Substituting for dX from equation (3.3) we obtain

$$dG = \frac{\partial G}{\partial X}(adt + bdZ) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2dt$$

Collecting the terms in dt we obtain

$$dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}bdZ \tag{3.20}$$

in this case the drift rate is $\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2$ and the variance is $\left(\frac{\partial G}{\partial X}\right)^2 b^2$

Multidimensional Itô's Lemma

Occasionally functions may have more than one random variable. In this case we refer to equation (1.3) from which we can get a family of differential equations using models for different underlying assets as

$$dX_i = \mu_i X_i dt + \sigma_i X_i dZ_i, \tag{3.21}$$

where X_i is the Stock price of the i^{th} asset,$i = 1, \ldots\ldots, N$, and μ_i and σ_i the drift and volatility of the i^{th} asset respectively, while dZ_i is the increase of the Wiener process of the i^{th} asset. We have dZ_i is equal to $\epsilon_i\sqrt{dt}$ where ϵ_i is a random number drawn from the normal distribution table. Thus dZ_i has a mean of zero and a standard deviation of $\sqrt{dt}$ hence

$$E(dZ_i) = 0 \text{ and } E(dZ_i^2) = dt$$

If Z_i and Z_j are correlated, the Wiener processes dZ_i and dZ_j, where $\text{Var}(dZ_i, dZ_j) = E(dZ_i dZ_j = \rho_{ij} dt)$, in this case ρ_{ij} is correlation coefficient between the i^{th} and j^{th} Wiener processes.

To be able to manipulate functions of many stochastic variables we need the multidimensional Itô's lemma.
In general we can consider a function $G(X_1, X_2, \ldots\ldots X_N, t)$ of stochastic variables $X_1, X_2, \ldots\ldots\ldots X_N$ and t,

then by Itô's lemma we have

$$dG = \left(\frac{\partial G}{\partial t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\rho_{ij}X_iX_j\frac{\partial^2 G}{\partial X_i \partial X_j}\right)dt + \sum_{i=1}^{N}\frac{\partial G}{\partial X_i}dX_i \tag{3.22}$$

where $dZ_i^2 = dt$, $dZ_j^2 = dt$ and $dZ_i dZ_j = \rho_{ij}dt$, [17],[37],[56][57].

By Itô's multiplication table we have:-

*	dZ_i	dt
dZ_j	$\rho_{ij}dt$	0
dt	0	0

In a case of two random variables X_1 and X_2 and a deterministic variable t, that is,

$$dX_1 = m_1(X_1, X_2, t)dt + n_1(X_1, X_2, t)dZ_1$$

and

$$dX_2 = m_2(X_1, X_2, t)dt + n_2(X_1, X_2, t)dZ_2$$

in which dZ_1 and dZ_2 are Brownian increments, both normally distributed with variance dt (since $dZ_i^2 = dZ_j^2 = dt$) and correlation coefficient ρ, $-1 \le \rho \le 1$, therefore from equation (3.22), we have[17],[37],[56][57][58],

$$dG = \left(\frac{\partial G}{\partial t} + \frac{1}{2}n_1^2\frac{\partial^2 G}{\partial X_1^2} + \frac{1}{2}n_2^2\frac{\partial^2 G}{\partial X_2^2} + \rho n_1 n_2 \frac{\partial^2 G}{\partial X_1 \partial X_2}\right)dt + \frac{\partial G}{\partial X_1}dX_1 + \frac{\partial G}{\partial X_2}dX_2 \tag{3.23}$$

3.3.1 Black Scholes Merton Partial Differential Equation

The following assumptions were made in deriving the Black-Scholes-Merton option price model [17].

1. The stock price follows a geometric Brownian motion with μ(drift rate) and σ(volatility) as constant

2. Short selling of assets with full use of proceeds is allowed.

3. There are no transaction costs or taxes.

4. The asset is perfectly divisible.

5. There are no dividends during the life of a derivative

6. There are no riskless arbitrage opportunities.

7. Asset trading is continuous.

8. The risk-free rate of interest, r, is constant throughout the trading time.

Suppose C is the price of a call option or other derivatives and is a function of S and t, twice differentiable in S and t, where S is the spot price of the asset at any time t . From assumption (1), the stock price follows the geometric Brownian motion given in equation (3.14), we have,

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0,$$

which is the Black Scholes Merton Partial Pifferential Equation.

3.4 The Nonlinear Black Scholes Merton Partial Differential Equation

3.4.1 Feedback effect of hedging in illiquid markets

In the traditional derivation of the the Black Scholes Merton Partial Differential Equation there is always an assumption that the replication trading strategy has no influence on the price of the underlying asset itself, that the asset price moves randomly. This could be justified by random flow of information concerning the asset and the economy specifically applicable to large investors. It is therefore necessary to asses the influence of these trading strategies on the price of the underlying and thus in the feedback loop.

The following strategies are used by Portfolio managers of large investments which may cause replication. This strategy known as portfolio insurance is popular in the European Put option. Any simple options having values $C(S,t)$ can be replicated by holding $\delta(S,t) = \frac{\partial C}{\partial S}(S,t)$ shares at a time if the share price is S. We can now consider $\Delta(S,t)$ to be a corresponding trading strategy. Put replication is one of the trading strategies that correspond to one example of portfolio insurance which we ought to analyze. Let us assume that the excess demand takes the form $\chi(S,t,x)$ being a function of price S, time t and a random influence x. The random influence ensures that our model does not stray too far. We regard such influence as the effect of new information that is arriving at random on the value of the underlying new asset or the action of noise traders.

Generally $\frac{\partial \chi}{\partial S}$ is negative, rising price leads to falling demand. At any given time the equilibrium price S^* is the price for which demand equals to the supply or generally the excess demand equals to zero ($\chi(S^*, t, x) = 0$) Any typical market would therefore return to equilibrium after undergoing disturbance due to the forces of supply and demand.([3],[12],[13],[57]) There exists a possibility of disequilibrium but speed of information flow and sufficient numbers of professionals on the stock markets guarantee full equilibrium in stocks, hence flows in Modern markets is a good approximation. In the nonlinear Black scholes model therefore both demand and supply can change because of the stochastic nature of the parameter. We therefore add extra demand resulting from hedging the put option to the original demand. With this additional demand the equilibrium becomes

$$S^* + \Delta(S, t) = 0 \tag{3.24}$$

Apart from demand arising from noise we shall also have a completely deterministic demand due to the trading strategy Δ. With an additional demand of the form $\Delta(S, t)$, the equilibrium condition equation $\chi(S^*, t, x) = 0$ becomes $\chi(S, t, x) + \Delta(S, t) = 0$ which must also hold for the change in χ and Δ given as

$$d\chi = d\Delta = 0$$

For simplicity we could consider an arbitrary excess demand function χ assuming now that $\chi(S, t, x) = \frac{1}{\varepsilon}(x - S)$ where ε is a positive real number and x follows a stochastic process in equation

$$dx = \mu_x(S, t)dt + \sigma_x(S, t)dZ,$$

where μ_x and σ_x can be functions of x and t, hence x is an intrinsic value. The parameter ε shows how strong the excess demand function reacts to change in price. If the price changes by dS the excess demand changes by $-\frac{ds}{\varepsilon}$. For liquid markets ε is small while for illiquid markets it is large.[3],[12],[13],[57]

With the appropriate choice of scaling under undisturbed equilibrium, ε^{-1} is also equal to the price elasticity of demand and applying Itô's lemma to $S + \varepsilon\Delta(S,t) = 0$ the stochastic process followed by S is

$$dS = \mu_S(S,t)dt + \sigma_S(S,t)dX$$

with μ_S and σ_S given by

$$\mu_S = \frac{\sigma_S}{\sigma x}\left(\mu_S + \varepsilon\left(\frac{\partial\Delta}{\partial t} + \frac{1}{2}\sigma_S^2\frac{\partial^2\Delta}{\partial S^2}\right)\right)$$

and

$$\sigma_S = \frac{\sigma_x}{1 - \varepsilon\frac{\partial\Delta}{\partial S}}$$

[3],[12],[13],[57]

3.4.2 The Nonlinear Model on illiquid markets

We could examine the drift μ_s and the variance σ_S of the modified price process. Both will have a term of the form $1 - \varepsilon\frac{\partial\Delta}{\partial S}$ in the denominator which the negative of the total demand function. When this becomes zero the demand function has a zero slope.

However when $\frac{\partial\Delta}{\partial S} < \varepsilon^{-1}$, a positive $\frac{\partial\Delta}{\partial S}$ will increase both μ_S and the absolute value of σ_S hence the market becomes volatile. Conversely if $\frac{\partial\Delta}{\partial S}$

is negative it will decrease the volatility of the market.

If Δ is regarded as the traders strategy to replicate a derivative security C, the relation $\Delta = \frac{\partial C}{\partial S}$ will yield a replication of the derivative security C with positive $\Gamma = \frac{\partial^2 C}{\partial S^2}$ which destabilizes the market of the underlying hence long positions in put and call options have positive gamma.The Nonlinear Black Scholes Merton Partial Differential Equation will therefore become

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{3.25}$$

and $\sigma(\Gamma) = [\sigma^{+}\text{if } \Gamma < 0]$ and $\sigma(\Gamma) = [\sigma^{-}\text{if } \Gamma > 0]$. Assuming the interest rate is equal to zero we have

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\sigma^2}{\left(1 - \varepsilon S\frac{\partial^2 C}{\partial S^2}\right)^2} S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = 0, \tag{3.26}$$

which is the Nonlinear Black Scholes Merton Partial Differential Equation.

3.4.3 Models with transaction costs

Effect of transaction costs

Transaction costs are incurred in the buying and selling of the underlying. The assumption held in deriving the classical Black Scholes formular that there are no transaction costs is therefore incorrect since in practice the costs do exist. Depending on the market the costs may or may not be important. However in a market with high transaction costs rehedging becomes too costly. In most liquid markets the costs are low hence it is possible to hedge quite often. The Black-Scholes model requires a continuous portfolio adjustment in order to hedge the position without any risks. The presence of transaction costs in the infinite number of transactions makes the process expensive. The hedger therefore needs to find a balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. This leads to "imperfect" hedging which culminates into the option being overpriced or underpriced to an extent where riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices instead.

In markets with transaction costs there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option. This necessitates an alternate relaxation of the hedging condition to better replicate the pay-off of derivative securities

The model of Leland

The modeling of transaction costs was initiated by Hayne Leland in 1985. This model is the first such model in finance and will therefore form a strong basis in this study. In using the Leland's method one hedges an option with a delta calculated in the same way as in the Black-Scholes delta, but with a modified (adjusted) volatility. This adjustment depends on the sign of the second derivative of the option price with respect to the price of the underlying. The Leland's approach minimizes the risk of the local risk of the hedging strategy. Leland's hedging with a modified hedging volatility " equalizes" the replication error across different stock paths which also reduces the total risk of a hedging strategy. The model can be extended to cover the cases of pricing and hedging an option portfolio on a commodity future contract, a portfolio of strong path-dependent option on a stock, and options on several assets. The Leland's approach yields a parabolic partial differential equation.

In markets assumed to have no transaction costs the option price is always equal to the cost of setting up the replicating portfolio. This follows from the absence of the arbitrage argument. Leland's idea was to include the expected transaction costs in the cost of a replicating portfolio, meaning that the price of an option must be equal to the expected cost of replicating portfolio including the transaction costs. This results into a situation where a market maker who writes a European call option for example and constructs a replicating portfolio, should sell it with a premium which offsets the expected transaction costs. On the contrast a market maker who buys a European call option and constructs a replicating portfolio, should buy the option with a discount to offset transaction costs. Leland further assumed that the revision of replicating portfolio must occur at fixed regular intervals of length δt we therefore present the Leland's model using the interpretation of Hoggard, Whalley and Wilmott. We consider a continuous time economy with a risky security, for example, a stock and a risk-free money market account which provides a constant rate of interest r. The price of the stock S, envelopes according to the stochastic differential equation (1.3)

$$dS = \mu S dt + \sigma S dZ$$

hence over a sufficiently small time interval δt the change in stock price is given by

$$\delta S = \mu S \delta t + \sigma S \epsilon \sqrt{\delta t} + o(\delta t^{\frac{3}{2}}), \qquad (3.27)$$

where ϵ is a random drawing from a normal distribution table.

We set up a hedged portfolio Π as

$$\Pi = C(S,t) - \Delta S \qquad (3.28)$$

where $\Delta = \frac{\partial C}{\partial S}(S,t)$. Henceforth we suppress dependance of Π, C, Δ on t over δt. After a given time δt therefore the portfolio becomes

$$\Pi + \delta\Pi = C(S+\delta S, t+\delta t) - \Delta(S+\delta S) \qquad (3.29)$$

from which it follows that

$$\delta\Pi = C(S+\delta S, t+\delta t) - \Delta(S+\delta S) - C(S,t) + \Delta S \qquad (3.30)$$

Expanding this in Taylor's series we obtain

$$\begin{aligned}\delta\Pi &\approx \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\delta S)^2 + \ldots - \Delta\delta S \\ &\approx \sqrt{\delta t}\sigma S\epsilon\left(\frac{\partial C}{\partial S} - \Delta\right) + \delta t\left(\frac{\partial C}{\partial t} + \mu S\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 \epsilon^2 \frac{\partial^2 C}{\partial S^2}\right) + \ldots\end{aligned}$$

which has not accounted for the inevitable transaction costs that will be incurred on rehedging. The costs are

$$kS|C|.$$

The quantity C of the underlying asset that are bought is given by the change in the delta from a given time step to the next:

$$C = \frac{\partial C}{\partial S}\Big(S + \delta S, t + \delta t\Big) - \frac{\partial C}{\partial S}(S, t)$$

which by Taylor's theorem is given by

$$C = \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2}\delta S + \frac{\partial^2 C}{\partial S \partial t}\delta t + \ldots - \frac{\partial C}{\partial S} = \frac{\partial^2 C}{\partial S^2}\delta S + \frac{\partial^2 C}{\partial S \partial t}\delta t$$

where all derivatives are now evaluated at(S, t). After two terms canceling we get the approximation

$$C \approx \frac{\partial^2 C}{\partial S^2}\delta S = \frac{\partial^2 C}{\partial S^2}\sigma S \epsilon \sqrt{\delta t}$$

Subtracting the cost from the change in portfolio value gives a total change of

$$\begin{aligned} d\Pi &= \delta\Pi - kS|C| = \sqrt{\delta t}\sigma S \epsilon\Big(\frac{\partial C}{\partial S} - \Delta\Big) + \delta t\Big(\frac{\partial C}{\partial t} + \mu S\Big(\frac{\partial C}{\partial S} - \Delta\Big) \\ &+ \frac{1}{2}\sigma^2 S^2 \epsilon^2 \frac{\partial^2 C}{\partial S^2}\Big) - k\sigma S^2 |\epsilon| \sqrt{\delta t}\left|\frac{\partial^2 C}{\partial S^2}\right| + \ldots \end{aligned}$$

The mean of this is

$$E[d\Pi] = \delta t\Big(\frac{\partial C}{\partial t} + \mu S\Big(\frac{\partial C}{\partial S} - \Delta\Big) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - k\sigma S^2 \sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\Big) + \ldots$$

because $E[\epsilon] = 0$, $E[\epsilon^2] = 1$ and $E[|\epsilon|] = \sqrt{\frac{2}{\pi}}$

We also find that

$$\begin{aligned} E[(d\Pi)^2] &= \delta t E\left[\sigma^2 S^2 \epsilon^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 - 2k\sigma S^2 \left|\frac{\partial^2 C}{\partial S^2}\right| \sigma S \left(\frac{\partial C}{\partial S} - \Delta\right)\epsilon|\epsilon| \right. \\ &+ \left. k^2 \sigma^2 S^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2 \epsilon^2 \right] + \ldots \\ &= \delta t \left(\sigma^2 S^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 + k^2 \sigma^2 S^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right) + \ldots \end{aligned}$$

since

$$E[\epsilon|\epsilon|] = 0.$$

The variance of the portfolio change is therefore

$$\begin{aligned} \text{var}[d\Pi] &= E[(d\Pi)^2] - (E[d\Pi])^2 \\ &= \delta t \left(\sigma^2 S^2 \left(\frac{\partial C}{\partial S} - \Delta\right)^2 + \left(1 - \frac{2}{\pi}\right) k^2 \sigma^2 S^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right) \end{aligned}$$

to leading order. For finite hedging period δt and finite cost k this cannot generally be made to vanish. However the variance, or risk, can be minimized by choosing

$$\Delta = \frac{\partial C}{\partial S}$$

with this choice,

$$E[d\Pi] = \delta t \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - k\sigma S^2 \sqrt{\frac{2\delta t}{\pi}} \left|\frac{\partial^2 C}{\partial S^2}\right|\right) + \ldots$$

to leading order. This quantity is an expectation allowing for the expected amount of transaction costs. We now set this quantity equal to the amount that would have been earned by a risk free account:

$$\delta t\left(\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}-k\sigma S^2\sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\right)$$

$$= r\Pi\delta t = r\left(C - S\frac{\partial C}{\partial S}\right)\delta t$$

On dividing by δt and rearranging we obtain the Hoggard-whalley-Wilmott representation of the Leland's model given by

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}-k\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 C}{\partial S^2}\right|+rS\frac{\partial C}{\partial S}-rC=0 \qquad (3.31)$$

3.5 Logistic models

3.5.1 Nature

Logistic differential equations have been found to give more accurate results than non logistic ones. Applied to population, Logistic models are based on the exponential growth and decay model, but they include an overcrowding term, or a nonconstant growth rate, that reflects the limitations on growth due to the scarcity of resources and living space.
In finance, a logistic equation for asset prices can be obtained by considering random responses in the forces of supply and demand during trading. This becomes possible when we introduce the excess demand function and apply it in the framework of the Walrasian (Walrasian-Samuelson) price adjustment mechanism[37].

3.5.2 The Law of Demand

The law of demand states that the quantity of a good or service is negatively related to its price,*ceteris paribus*. That is, consumers will purchase more of a good or service at a lower price than at a higher price. As price rises, *ceteris paribus*, a consumer will demand a smaller quantity of a good or service.

3.5.3 The Law Of Supply

This law states that a quantity supplied of a good or service is usually a positive function of price, *ceteris paribus*. That is suppliers will supply less of a good or a service at a lower price and as price rises the quantity supplied will increase.

3.5.4 Walrasian Equilibrium(Equilibrium Price)

This is a state of stability or balance where the quantity of a good or service supplied is equal to the quantity of the same good or service demanded.

3.5.5 Excess Demand

Of necessity, we note that in stock price modeling the price of an asset is assumed to respond to the excess demand which is the difference between the quantity of an asset demanded and the quantity of the same asset supplied as given by equation (1.4), That is,

$$EDS(t) = Q_D S(t) - Q_S s(t)$$

where:

$EDS(t)$ is the excess demand,$Q_D S(t)$and $Q_S S(t)$ are the quantities demanded and supplied respectively at a given time,t and price, S(t).

Just like in the predator-prey ecosystem where there is "give and take", the market structure with forces of supply and demand exhibit two forces in the market which affect each other striving to strike a balance called the market equilibrium.

This comparative phenomenon has made it possible to apply the idea of logistic equation, first used by verhlust(1838), and Reed(1920). In Verhlust's model for studying dynamics of human population growth in the United states, he took p^* to represent the environmental carrying capacity in which a population lives, which favorably compares to $s^*$the Walrasian equilibrium market price, a point where the quantity supplied and demanded in the market are equal.

This has led to the Verhlust logistic Black-Scholes-Merton partial differential equation with constant volatility. In this study we intend to formulate the Logistic Black-Scholes-Merton partial differential equation with stochastic volatility.

3.5.6 Logistic Geometric Brownian Motion Model

Using the Walrasian law and the excess demand function(1.4), the logistic geometric Brownian motion according to Onyango ([37],[38]) is of the form

$$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ \tag{3.32}$$

or

$$\frac{1}{S}\frac{dS}{(S^* - S)} = \mu dt + \sigma dZ \tag{3.33}$$

where S^* is the Walrasian market equilibrium price, S is the stock price at any given time t, μ is the drift rate and σ is the volatility of the stock price at any given time t. In this model volatility is constant.

Chapter 4

THE LOGISTIC LINEAR BLACK-SCHOLES-MERTON PARTIAL DIFFERENTIAL EQUATION

4.1 Introduction

In this chapter we use Itô's lemma given by equation (3.22) , the Logistic geometric Brownian motion, equation (3.32) and a stochastic volatility model, equation (3.9), to derive the Logistic Black-Scholes-Merton partial differential equation.

4.2 The Logistic Differential Equation

Suppose the price of an asset follows a Logistic Geometric Brownian Motion equation (3.32) then we have,

$$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ_1 \tag{4.1}$$

in which as already defined in section 3.5, S^* is the Walrasian equilibrium price while S is the asset price at a given time t, and the stochastic volatility model given as,

$$d\sigma = \mu_\sigma \sigma dt + v_\sigma \sigma dZ_2 \tag{4.2}$$

where σ is the asset price volatility, μ_σ and v_σ are the mean and variance of asset volatility respectively, whereas dZ_1 and dZ_2 are the Wiener processes associated with the two differential equations (4.1) and (4.2) respectively. We can also let the Wiener processes have a correlation ρ. Considering equations (4.1) and equation (4.2), the value of an option is therefore a function of three variables, $C(S, \sigma, t)$, where C is the price of the call option and S is the asset price. Since volatility is not a traded asset, its randomness cannot be easily traded away. Having two other sources of randomness therefore, we need to hedge our options against two other contracts, one being the underlying asset as usual but the other to hedge the volatility risk. We therefore set up a portfolio as,

$$\Pi = C - \delta S - \delta_1 C_1 \tag{4.3}$$

The change in the portfolio $d\Pi$ will be given by,

$$d\Pi = dC - \delta dS - \delta_1 dC_1 \tag{4.4}$$

Using Itô's lemma on S, σ and t as defined in section (3.3) by equation (3.22), we can use the application in equation (3.23) on equations (4.1) and equation (4.2) to obtain the change in portfolio in a time dt as,

$$\begin{aligned} d\Pi &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2}\right) dt \\ &+ \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial \sigma} d\sigma - \delta dS \\ &- \delta_1 \left(\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C_1}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C_1}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C_1}{\partial \sigma^2}\right) dt \\ &- \delta_1 \frac{\partial C_1}{\partial S} dS - \delta_1 \frac{\partial C_1}{\partial \sigma} d\sigma \end{aligned} \tag{4.5}$$

Collecting the terms in dS and $d\sigma$ in equation (4.5) we obtain

$$\begin{aligned} d\Pi &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2}\right) dt \\ &- \delta_1 \left(\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C_1}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C_1}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C_1}{\partial \sigma^2}\right) dt \\ &+ \left(\frac{\partial C}{\partial S} - \delta_1 \frac{\partial C_1}{\partial S} - \delta\right) dS + \left(\frac{\partial C}{\partial \sigma} - \delta_1 \frac{\partial C_1}{\partial \sigma}\right) d\sigma \end{aligned} \tag{4.6}$$

In order to eliminate all randomness we choose $\frac{\partial C}{\partial S} = \delta_1 \frac{\partial C_1}{\partial S} + \delta$ and $\frac{\partial C}{\partial \sigma} = \delta_1 \frac{\partial C_1}{\partial \sigma}$ making terms involving dS and $d\sigma$ to be equal to zero. After eliminating dS and $d\sigma$ which contain the Wiener processes dZ_1 and dZ_2

respectively, equation (4.6) becomes a non stochastic differential equation,

$$\begin{aligned} d\Pi &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma\frac{\partial^2 C}{\partial S\partial\sigma} + \frac{1}{2}v_\sigma^2\sigma^2\frac{\partial^2 C}{\partial\sigma^2}\right)dt \\ &- \delta_1\left(\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C_1}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma\frac{\partial^2 C_1}{\partial S\partial\sigma} + \frac{1}{2}v_\sigma^2\sigma^2\frac{\partial^2 C_1}{\partial\sigma^2}\right)dt \end{aligned} \tag{4.7}$$

We can use no arbitrage argument to set the return of the portfolio to be equal to the risk free interest rate r as

$$d\Pi = r\Pi dt \tag{4.8}$$

Substituting equations (4.3) and (4.7) into equation (4.8) we obtain,

$$\begin{aligned} &\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma\frac{\partial^2 C}{\partial S\partial\sigma} + \frac{1}{2}v_\sigma^2\sigma^2\frac{\partial^2 C}{\partial\sigma^2}\right)dt \\ &- \delta_1\left(\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C_1}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma\frac{\partial^2 C_1}{\partial S\partial\sigma} + \frac{1}{2}v_\sigma^2\sigma^2\frac{\partial^2 C_1}{\partial\sigma^2}\right)dt \\ &= r(C - \delta S - \delta_1 C_1)dt \end{aligned} \tag{4.9}$$

We now have a situation where we have one equation with two unknowns C and C_1. Given that $\delta = \frac{\partial C}{\partial S}$ and $\delta_1 = \frac{\partial C_1}{\partial S}$ and that both are affected by a hedge ratio $\frac{\partial C}{\partial\sigma}$ and $\frac{\partial C_1}{\partial\sigma}$ respectively (which are also the sensitivity of option price to volatility) respectively, we can collect terms in C on one side and those in C_1 to be on the other to obtain,

$$\frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2} + rS\frac{\partial C}{\partial S} - rC}{\frac{\partial C}{\partial \sigma}}$$

$$= \frac{\delta_1\left(\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C_1}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C_1}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C_1}{\partial \sigma^2} - rC_1\right)}{\frac{\partial C_1}{\partial \sigma}}$$

Since the two different options will have different payoffs, this possibility can only be obtained if the left hand side and the right hand side are independent of the contract type. Both sides therefore can only be functions of the independent variables, S, σ and t and thus we have

$$\begin{aligned}&\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2} \\ &+ rS\frac{\partial C}{\partial S} - rC = -(\mu_\sigma - \lambda v_\sigma)\frac{\partial C}{\partial \sigma}\end{aligned} \tag{4.10}$$

for some function $\lambda(S, \sigma, t)$ which is the market price of volatility risk and $\mu_\sigma - \lambda v_\sigma$ is the risk neutral drift rate of volatility. Rewriting this equation (4.10) we obtain

$$\begin{aligned}&\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2} \\ &+ rS\frac{\partial C}{\partial S} + (\mu_\sigma - \lambda v_\sigma)\frac{\partial C}{\partial \sigma} - rC = 0\end{aligned} \tag{4.11}$$

Equation (4.11) gives us the equivalent of the Logistic Black-Scholes-Merton partial differential equation with stochastic volatility.

If Z_1 and Z_2 are of the same distribution then $dZ_1 = dZ_2$ which implies that $\rho = 1$ since $dZ_1^2 = dt$ hence equation (4.11) becomes

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C}{\partial S^2} + \sigma^2 S(S^* - S)v_\sigma \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}v_\sigma^2 \sigma^2 \frac{\partial^2 C}{\partial \sigma^2}$$
$$+ rS\frac{\partial C}{\partial S} + (\mu_\sigma - \lambda v_\sigma)\frac{\partial C}{\partial \sigma} - rC = 0 \qquad (4.12)$$

Equation (4.11) is therefore the Logistic Black-Scholes-Merton partial differential equation with stochastic volatility.

4.3 Deterministic Price Adjustment Model.

To make the price adjustment more computational, we begin by taking supply and demand functions to be fixed functions of instantaneous price $S(t)$. Then at equilibrium asset price point, P^*, demand $Q_D(S^*)$ is equal to supply, $Q_S(S^*)$. On the assumption of fixed supply and demand curves, S^* is constant. Away from equilibrium, excess demand for the security will raise its price, S, and an excess supply will lower its price. Thus the sign of the rate of change of price, S, with respect to time, t, will depend on the sign of the excess demand. If we linearise $Q_D(S(t))$ and $Q_S(S(t))$ about the constant equilibrium price S^*, the deterministic model of price adjustment becomes

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = k(\alpha + \beta)(S^* - S(t)) \qquad (4.13)$$

where $Q_D(S(t)) = \alpha(S^* - S(t))$, $Q_S(S(t)) = -\beta(S^* - S(t))$, and constants α and β are demand and supply sensitivities respectively. Putting $r =$

$k(\alpha + \beta)$in equation (4.13), we get the deterministic logistic equation

$$\frac{dS(t)}{dt} = rS(t)(S^* - S(t)) \tag{4.14}$$

see [29],[37],[38].

This is a deterministic logistic (first- order) ordinary differential equation in $S(t)$. Thus the fractional growth of $S(t)$ is linear in $S(t)$. This contrasts with exponential growth, where the fractional growth is constant (independent of $S(t)$).

The logistic equation was first investigated by Pierre-Francois Verhulst in [53], as an improvement on the Malthusian model of population dynamics, hence it is also known as Verhulst-logistic differential equation. Since then it has been applied in several areas.

The solution set of equation (4.14) is given by

$$S(t) = \frac{S^* S(0)}{S(0) + (S^* - S(0))e^{-rS^* t}} \tag{4.15}$$

where $S(0)$ is a parameter interpreted as the initial price an asset. From equation (4.15) we observe that as $t \to \infty$, the term $S(t) \to \frac{S^* S(0)}{S(0)} = S^*$ The asset price thus settles into a constant level, called a steady state or equilibrium, at which no further change will occur.

4.4 Logistic price adjusment model

In this section we model random fluctuations in supply and demand by changes $\delta\alpha$, $\delta\beta$ in the respective sensitivities. Consider that, $Q_D(S(t))$ and $Q_(S(t))$ to represent averaged effects of supply and demand respectively, and suppose that both curves steepen or level off in response to random observed trades: cumulatively they execute a random walk or Wiener diffusion process. From equation (4.13) we have

$$\frac{dS(t)}{S(t)dt} = k(\alpha+\beta)(S^* - S(t)) + k(\delta\alpha+\delta\beta)(S^* - S(t))$$

or

$$\frac{dS(t)}{S(t)(S^* - S(t))} = k(\alpha+\beta)dt + k(\delta\alpha+\delta\beta)dt \tag{4.16}$$

From equation (4.16), we may put $\mu = k(\alpha+\beta)$ (logistic growth parameter) and $\sigma dZ = k(\delta\alpha+\sigma\beta)dt$(noise process) to obtain

$$\frac{dS(t)}{S(t)(S^* - S(t))} = \mu dt + \sigma dZ \tag{4.17}$$

Equation (4.16) defines an Itô process evolving according to the stochastic differential equation (3.32) of the form

$$dS(t) = \mu S(t)(S^* - S(t))dt + \sigma S(t)(S^* - S(t))dZ \tag{4.18}$$

We refer to equation (4.17) as Logistic Price Adjustment Model (LPAM model), or Verhulst-Price adjustment model (VPAM),see [37]. In the risk-less case ($\sigma = 0$), equation (4.17) reduces to the logistic equation

(4.14) with equation (4.16). Using Itô's lemma, the solution of (4.17) is expressed as

$$ln\left(\frac{S(t)}{|S^* - S(t)|}\right) = ln\left(\frac{S(0)}{|S^* - S(0)|}\right) + \mu S^*(t - t_0) + \sigma S^* Z(t) \quad (4.19)$$

Re-arranging and simplifying equation (4.19), we get

$$S(t) = \frac{S^* S(0)}{S(0) + (S^* - S(0))e^{-(\mu S^*(t-t_0)+\sigma S^* Z(t))}} \quad (4.20)$$

This price dynamics is referred to as logistic Brownian motion of asset price, $S(t)$. When $\sigma = 0$, then we get the deterministic logistic equation (4.15)

4.5 Partial differential equation for logistic Price Adjustment Model.

In this section we derive the partial differential equation for logistic price adjustment model in a case where volatility is constant. Let $C(S,t)$ be the option value depending on asset price, S and time t, then by Itô's lemma (3.22)we have

$$dC\Big(S(t),t\Big) = \frac{\partial C(S,t)}{\partial t}dt + \frac{\partial C(S,t)}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}dS^2 \quad (4.21)$$

For logistic Brownian motion we have

$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ$

and

$dS^2 = \sigma^2 S^2 (S^* - S)^2 dt$

Substituting in equation (4.21) and simplifying we get

$$dC\Big(S(t),t\Big) = \left(\mu S(S^* - S)\frac{\partial C(S,t)}{\partial S} + \frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 C(S,t)}{\partial S^2}\right) dt + \sigma S(S^* - S)\frac{\partial C(S,t)}{\partial S} dZ \quad (4.22)$$

By using the no-arbitrage argument(4.8), which implies that the percentage return of the portfolio over the time interval dt should equal the risk-free interest rate, r. That is

$$d\pi(t)_{risk-free} = r\pi(t)_{risk-free} \quad (4.23)$$

Thus we get

$$\left(\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 (S^* - S)^2 \frac{\partial^2 C(S,t)}{\partial S^2}\right) dt = r\left(C(S,t) - \frac{\partial C(S,t)}{\partial S} S\right) dt \quad (4.24)$$

Further simplification of (4.24) yields a partial differential equation given by

$$\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 (S^* - S)^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0 \quad (4.25)$$

This modified Black-Scholes-Merton partial differential equation is a logistic partial differential equation.

Chapter 5

THE LOGISTIC NONLINEAR BLACK-SCHOLES-MERTON PARTIAL DIFFERENTIAL EQUATION

5.1 Introduction

In this chapter we use the Geometric Brownian motion equation (2.4) and the random walk in discrete time given by

$$\delta S = \mu S \delta t + \sigma S \epsilon \sqrt{\delta t} + o(\delta t^{\frac{3}{2}}) \tag{5.1}$$

with the assumption that the portfolio is revised every δt, where δt is a finite and a fixed time step and that the hedged portfolio has an expected return equal to that from a risk free bank deposit, which is the same as the valuation policy in discrete hedging with no transaction costs.

5.2 Logistic nonlinear Black Scholes Merton partial differential equation

Suppose the price of an asset follows a Logistic Geometric Brownian motion given by equation (3.32) given as

$$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ,$$

then over a sufficiently small time interval δt the change in stock price is given by

$$\delta S = \mu S(S^* - S)\delta t + \sigma S(S^* - S)\epsilon\sqrt{\delta t} + o(\delta t^{\frac{3}{2}}) \tag{5.2}$$

where ϵ is a random drawing from a normal distribution table.
We set up a hedged portfolio Π as

$$\Pi = C(S,t) - \Delta S \tag{5.3}$$

where $\Delta = \frac{\partial C}{\partial S}(S,t)$. Henceforth we suppress dependance of Π, C, Δ on t over δt. After a given time δt therefore the portfolio becomes

$$\Pi + \delta\Pi = C(S + \delta S, t + \delta t) - \Delta(S + \delta S) \tag{5.4}$$

from which it follows that

$$\delta\Pi = C(S + \delta S, t + \delta t) - \Delta(S + \delta S) - C(S, t) + \Delta S \tag{5.5}$$

Expanding this in Taylor's series we obtain

$$\begin{aligned}\delta\Pi &= \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\delta S)^2 + \ldots - \Delta\delta S \\ &= \sqrt{\delta t}\sigma S(S^* - S)\epsilon\left(\frac{\partial C}{\partial S} - \Delta\right) + \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\epsilon^2\frac{\partial^2 C}{\partial S^2}\right) + \ldots\end{aligned}$$

which has not accounted for the inevitable transaction costs that will be incurred on rehedging. The costs are

$$kS|C|.$$

The quantity C of the underlying asset that are bought is given by the change in the delta from a given time step to the next:

$$C \approx \frac{\partial C}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial C}{\partial S}(S, t)$$

which can be approximated by

$$C = \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2}\delta S + \frac{\partial^2 C}{\partial S \partial t}\delta t + \ldots - \frac{\partial C}{\partial S}$$

where all derivatives are now evaluated at(S, t). After two terms canceling we get the approximation

$$C \approx \frac{\partial^2 C}{\partial S^2}\delta S \approx \frac{\partial^2 C}{\partial S^2}S\epsilon\sigma\sqrt{\delta t}$$

Subtracting the cost from the change in portfolio value gives a total change of $\delta\Pi = d\Pi - kS|\epsilon|$ which is

$$\begin{aligned}\delta\Pi &= \sqrt{\delta t}\sigma S(S^* - S)\epsilon\left(\frac{\partial C}{\partial S} - \Delta\right) + \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\epsilon^2\frac{\partial^2 C}{\partial S^2}\right) \\ &- k\sigma S^2|\epsilon|\sqrt{\delta t}\left|\frac{\partial^2 C}{\partial S^2}\right| + \ldots\end{aligned}$$

The mean of this is

$$E[\delta\Pi] = \delta t\left(\frac{\partial C}{\partial t} + \mu S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right) - k\sigma S^2\sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\right) + \ldots$$

Because $E[\epsilon] = 0$, $E[\epsilon^2] = 1$ and $E[|\epsilon|] = \sqrt{\frac{2}{\pi}}$
We also find that

$$\begin{aligned}E[(\delta\Pi)^2] &= \delta t E\left[\sigma^2 S^2(S^* - S)^2\epsilon^2\left(\frac{\partial C}{\partial S} - \Delta\right)^2 - 2k\sigma S^2(S^* - S)^2\left|\frac{\partial^2 C}{\partial S^2}\right|\sigma S(S^* - S)\left(\frac{\partial C}{\partial S} - \Delta\right)\epsilon|\epsilon|\right. \\ &+ \left. k^2\sigma^2 S^4(S^* - S)^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\epsilon^2\right] + \ldots \\ &= \delta t\left(\sigma^2 S^2(S^* - S)^2\left(\frac{\partial C}{\partial S} - \Delta\right)^2 + k^2\sigma^2 S^4(S^* - S)^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\right) + \ldots\end{aligned}$$

since

$$E[\epsilon|\epsilon|] = 0.$$

The variance of the portfolio change is therefore

$$\begin{aligned}var[\delta\Pi] &= E[(\delta\Pi)^2] - (E[\delta\pi])^2 \\ &= \delta t\left(\sigma^2 S^2(S^* - S)^2\left(\frac{\partial C}{\partial S} - \Delta\right)^2 + \left(1 - \frac{2}{\pi}\right)k^2\sigma^2 S^4(S^* - S)^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\right)\end{aligned}$$

to leading order. For finite hedging period δt and finite cost k this cannot generally be made to vanish. However the variance, or risk, can be minimized by choosing

$$\Delta = \frac{\partial C}{\partial S}$$

with this choice,

$$E[\delta\Pi] = \delta t\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C}{\partial S^2} - k\sigma S^2(S^* - S)^2\sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|\right)$$

to leading order. This quantity is an expectation allowing for the expected amount of transaction costs. We now set this quantity equal to the amount that would have been earned by a risk free account:

$$\delta t\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C}{\partial S^2}\right) - k\sigma S^2(S^* - S)^2\sqrt{\frac{2\delta t}{\pi}}\left|\frac{\partial^2 C}{\partial S^2}\right|$$

$$= r\Pi\delta t = r\left(C - S\frac{\partial C}{\partial S}\right)\delta t$$

On dividing by δt and rearranging we obtain the Logistic nonlinear Black Scholes Merton partial differential equation given by

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2(S^* - S)^2\frac{\partial^2 C}{\partial S^2} - k\sigma S^2(S^* - S)^2\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 C}{\partial S^2}\right| + rS\frac{\partial C}{\partial S} - rC = 0 \tag{5.6}$$

5.3 Conclusion and recommendation

In this thesis we have managed to derive a Logistic nonlinear Black Scholes Merton Partial differential equation based on the model with transaction costs which is appearing in this thesis for the first time in literature. This comes as a breakthrough in the study of the nonlinear Black Scholes Merton Partial differential equation and in its application in the prediction of future asset prices where transaction costs are considered together with the logistic geometric Brownian motion unlike in previous studies where the Brownian motion has been used.

We recommend that interested scholars solve the differential equation in order to enhance prediction of future asset prices.

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