CHARACTERIZATION OF SCALAR TYPE OPERATORS

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Abstract

A closed densely defined operator H, on a Banach space X, whose spectrum is contained in $\mathbb{R}$ and satisfies the growth condition $\| (z-H)^{-1} \| \leq C\frac{\langle z\rangle^{\alpha}}{|Iz|^{\alpha+1}}$ $\forall z \in i\mathbb{R}$ for some $\alpha \geq 0$ and $C > 0$ is of $(\alpha, \alpha+1)$ type $\mathbb{R}$. In studying spectral theory, the main interest has been finding a criteria for an operator to be of scalar type. Different approaches have been used so far, for example, Kantorovitz established a criteria using boundedness of operators with real spectrum acting on a reflexive Banach space. He also characterized scalar type operators using semigroup theory where it is shown that a bounded operator H is a scalar type if and only if iH generates a definite group. The method uses Laplace transform and mainly applies to bounded operators. Thus there is need to extend this characterization to unbounded operators. If $\alpha > 0$, $f \in \mathcal{U}$ where $\mathcal{U}$ is an algebra of smooth functions and H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$, then the integral $f(H) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}dxdy$ is norm convergent and defines an operator in $B(X)$ with $\| f(H) \| \leq c \| f \|_{n+1}$. The map $f \to f(H)$ is a $\mathcal{U}$- functional calculus for H. Using this functional calculus and the semigroup of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators, we characterize scalar type operators H satisfying the first inequality. We determine the necessary condition for a densely defined closed linear operator H acting on a Hilbert space $\mathcal{H}$ to be of scalar type for f in the algebra of smooth functions $\mathcal{U}$. We also characterize the scalar type operators using the semigroup theory of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators and then give some applications of scalar type operators in decomposability and abstract Cauchy problems with appropriate boundary conditions. This functional calculus is important since it applies to both bounded and unbounded operators.

Chapter 1

Introduction

1.1 Background

Spectral theory generally deals with the study of spectral values and the associated eigenvectors. One can study the spectral properties of operators on Banach spaces, for example the compact operators whose spectral properties are similar to that of matrices. The richest functional calculus occurs in the well known setting of self adjoint operators on a Hilbert space, which by the spectral theorem, has a functional calculus for any Borel measurable function on the spectrum of the operator. Let $f(z)$ be a holomorphic function where z is a complex number, then one can construct the functional calculus $f(T)$ where T is an operator. In other words, $f(T)$ a an extension from the complex function argument to an operator argument. If T is an $n \times n$ matrix with complex entries, then one can define $f(T)$ depending on the nature of f, for example if $p(z) = \sum_{i>0}^{m} a_i z^i$, then the associated polynomial functional calculus is defined as $p(T) = \sum_{i>0}^{m} a_i T^i$ where $T^\circ = I$ (Identity matrix). It defines a homomorphism from the ring of polynomial to the ring of $n \times n$ of matrices. Generally, for $f(T)$ to make sense, f need to be defined on the spectrum of T. In the case where the operator T denotes a matrix, then the corresponding spectral values tells us more to what extent $f(T)$ can be defined, i.e $f(\lambda)$ must be defined for all spectral value λ of T. For a general bounded operator, f must be defined on the spectrum of T. The functional calculus for bounded operators can be defined via Cauchy integral formula, that is,

$$f(T) = \frac{1}{2\pi i}\int_{\Gamma} f(\xi)(\xi - T)^{-1} d\xi$$

where $(\xi - T)^{-1}$ is the resolvent of T at ξ and the mapping $\xi \to (\xi - T)^{-1}$ is denoted by $R(\xi, T)$. The resolvent mapping is important in determination of the properties required of a functional calculus.

Now, let X, Y denote Banach spaces . We shall also assume that the underlying field is the complex field. Let $\mathbb{C}$, $\mathbb{R}$ denote complex and real fields respectively, while $\mathbb{Z}$ denotes the set of integers. We shall denote arbitrary complex numbers by $z := x + iy$ or $w := u + iv$ $x, y, u, v \in \mathbb{R}$ and $i := \sqrt{-1}$. $\mathbb{N}_o := \{0, 1, 2, ...\} = \mathbb{N} \cup \{0\}$, will denote the non-negative integers and $B(X, Y)$ denotes the space of all bounded linear operators from the linear space X to linear space Y and $B(X) := B(X, X)$. An operator H on X is said to be closed if its graph

$$G(H) := \{(f, Hf) : f \in D(H)\}$$

is closed. The graph of H is a closed subspace of $X \times X$ or equivalently, $f_n \in D(H)$, $g \in R(H)$ such that $f_n \to f$ in X and $Hf_n \to g$ implies $f \in D(H)$ and $Hf = g$. An operator H on X is said to be densely defined if its domain $D(H)$ is dense in X. Let H be an operator on X, then the resolvent set of H is defined as

$$\rho(H) := \{z \in \mathbb{C} : zI - H : D(H) \to X \, is \; invertible\}$$

Equivalently, $\rho(H)$ is the set of all $z \in \mathbb{C}$ such that $zI - H$ is bijective and $(zI - H)^{-1} \in B(X)$.

Theorem 1.1.1 (Closed Graph Theorem)

If H is a closed operator on X and domain $D(H)$ is a closed subspace of X, then H is bounded.

If H is closed, then we write $z - H$ for $zI - H$ and $R(z, H) := (z - H)^{-1}$ is called the resolvent operator of H or simply the resolvent of H. The set $\sigma(H) := \mathbb{C} \setminus \rho(H)$ is called the spectrum of H.

$R(z, H)$ is a norm holomorphic function of z and satisfies the resolvent identities.

$$R(z, H) - R(w, H) = -(z - w)R(z, H)R(w, H) \tag{1.1}$$

$$R(z, H)R(w, H) = R(w, H)R(z, H) \tag{1.2}$$

$$\frac{d^n}{dz^n}R(z, H) = (-1)^n R(z, H)^{n+1} \tag{1.3}$$

In this thesis, we consider a closed densely defined operator H acting on a Banach space X, and whose spectrum is contained in $\mathbb{R}$ and there exists a $C > 0$ such that inequality

$$\| (z - H)^{-1} \| \leq C \frac{\langle z \rangle^{\alpha}}{|Iz|^{\alpha+1}} \tag{1.4}$$

is satisfied for all $z \in i\mathbb{R}$ and some $\alpha \geq 0$, then H is referred to as $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator, see [29].

Here, $\langle z \rangle := \sqrt{1+ \mid z \mid^2}$ and Iz denotes the imaginary part of z. A special case is a Hermitian operator on a Hilbert space. If X^{**} is the second dual of X, we define a mapping $T : x \to F(x)$ of X into X^{**} where $Fx(f) = f(x)$, $f \in X^*$(dual of X). If this map is surjective, then X is called reflexive, that is, if X is a reflexive Banach space, then an operator $T \in B(X)$ is scalar type if it admits an integral representation with respect to a countably additive projection measure or equivalently if it admits a $C(\sigma(T))$ functional calculus where $\sigma(T)$ denotes the spectrum of T [8]. In particular, if T acts on a Hilbert space $\mathcal{H}$, then T admits $C(\mathbb{R})$ functional calculus if it is Hermitian. Generally, an operator T acting on a reflexive Banach space is scalar type if and only if it has a $C_o(\mathbb{R})$ functional calculus [19]. Here the space $C_o(\mathbb{R})$ denotes the space of continuous functions with compact support. According to Kantorovitz [20], if T is an operator with $\sigma(T) \subset \mathbb{R}$ and acting on a reflexive Banach space X, then T is scalar type if and only if its infinitesimal generator generates a uniformly bounded strongly continuous group.

1.2 Statement of the Problem

Our interest is to apply the properties of $(\alpha, \alpha+1)$ type $\mathbb{R}$ type operators, the $\mathcal{U}$ - functional calculus, and the semigroup theory to characterize the scalar type operators for some $\alpha \geq 0$ and give some applications to decomposability and abstract Cauchy problems with appropriate boundary conditions.

1.3 Objective of the study

The main objective of this study was to characterize the properties of scalar type operators. The specific objectives of this study includes;

1. To determine necessary conditions for a densely defined closed linear operator H acting on a Hilbert space space $\mathcal{H}$ to be of scalar type for f in the algebra of smooth functions $\mathcal{U}$.
2. To Characterize the scalar type operators using the semi-group theory and the functional calculus for $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators
3. To apply the theory of scalar type operators in decomposability of operators and some abstract Cauchy problems with appropriate boundary conditions.

1.4 Significance of the study

So far other approaches like Riez-Dunford functional calculus which relies more on the resolution of the identity have been used to show that an operator is of scalar type. Generally, their characterization have been based mainly on boundedness of the resolution of the identity and the growth of the resolvents. Our approach however is unique in that we have used the properties of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators and the $\mathcal{U}$-functional calculus to characterize the scalar type operator H and we have also given some applications in decomposability and abstract Cauchy equations with appropriate boundary conditions. This we believe has greatly contributed knowledge in the field of scalar type operators.

1.5 Research Methodology

We first reviewed the available literature on the spectral theory in general. We then deeply surveyed the existing literature on the scalar type operators and analyzed some of the methods which have so far been used to characterize the scalar type operators. The general knowledge of the holomorphic functional calculus, the semigroup theory of operator, the $\mathcal{U}$- functional calculus and the properties of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators for $\alpha \geq 0$ were also useful in solving our problems.

1.6 Organization of the study

In chapter one, we give the basic concepts and highlight some preliminary results necessary in facilitating the development of other chapters Chapter two looks at the literature review on the fundamental results in scalar type operators. The main items under consideration here includes spectral theorem for self adjoint operators, almost analytic functional calculus and applications of scalar type operators in decomposability and some abstract Cauchy equations with appropriate boundary conditions. In chapter three, we state and prove some basic properties of scalar type operators which will later helps us in the characterization of the $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators. In chapter four, tools developed in chapter two and three are used to characterize $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators. In particular, we characterize the scalar type operators using the $\mathcal{U}$ functional calculus and semigroup theory of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators. In chapter five, we look at the applications of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators in decomposability and to analyze some abstract Cauchy equations with appropriate boundary conditions. Finally in chapter six, we give the summary and recommendations for further research.

1.7 Basic concepts

1.7.1 Scalar operators and scalar type operators

Let X be a Banach space and $T \in B(X)$, then T is a spectral operator if it admits a representation of the form $T = S + Q$ where $Q \in B(X)$ is quasi-nilpotent operator commuting with T and $S \in B(X)$ is a scalar operator. An operator $S \in B(X)$ is a scalar operator if it can be represented by $S = \int_{\mathbb{C}} \lambda d\mu(\lambda)$ for some spectral measure $\lambda \in \mathbb{C}$. Type $\mathbb{R}$ operators are those that have their spectrum in $\mathbb{R}$, for instance scalar type $\mathbb{R}$ operators are scalar operators whose spectrum is contained in $\mathbb{R}$.

1.7.2 The $\mathcal{U}$ functional Calculus

The $\mathcal{U}$ functional calculus for an operator H of $(\alpha, \alpha + 1)$ type $\mathbb{R}$, is a continuous map $k : \mathcal{U} \to B(X)$ such that;

1. $k(fg) = k(f)k(g)$ for all $f, g \in \mathcal{U}$
2. If $w \in \mathbb{C} \setminus \mathbb{R}$, then $r_w \in \mathcal{U}$ and $k(r_w) := (w - H)^{-1}$

Here, $k(f) \equiv f(H)$

The definition of $f(H)$ for an operator H of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ for $f \in \mathcal{U}$ comes from the version of Helffer and Sjostrand [17] integral formula

$$f(H) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - H)^{-1} dxdy, \tag{1.5}$$

where $\tilde{f}$ is an analytic extension of f. One of the interesting developments in spectral theory recently has been the application of the new formula for a function $f(H)$ of a self adjoint operator H by Helffer-Sjostrand [17]. For unbounded self adjoint operators acting on a Hilbert space, it is proved in [17] that (1.5), is an alternative characterization of C_o-functional calculus. For our case, we shall apply the $\mathcal{U}$ functional calculus constructed on more general Banach space in [28]. It is shown in [28] that if $n > \alpha \geq 0$, $f \in \mathcal{U}$ and H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$, then the integral (1.5) is norm convergent and defines an operator in $B(X)$ with $\| f(H) \| \leq c_\alpha \| f \|_{n+1}$ for some $n \geq 1$ for some $c_\alpha > 0$. Well bounded operators are bounded operators with a functional calculus for absolutely continuous functions. Many differential operators, not necessarily on a Hilbert space, have well bounded resolvents, in contrast to situation for spectral operators of scalar type. However, most standard differential operators on $L^\infty(\mathbb{R})$ or $L^1(\mathbb{R})$ of the real line or the unit interval are not well bounded. We therefore resort to a more general functional calculus which in our case will be a $\mathcal{U}$ functional calculus. In this thesis, a $\mathcal{U}$ functional calculus is defined for a closed densely linear operator H on a Banach space X with $\sigma(H) \subseteq \mathbb{R}$ satisfying the resolvent estimate (1.4), and for functions from weighted Sobolev spaces [6]. The calculus used here is based on almost analytic extension to $\mathbb{C}$ of infinitely differentiable functions defined on $\mathbb{R}$ and the Helffer-Sjostrand formula [17]. One of the advantages of this formula is that it allows one to pass easily and flexibly from resolvent estimates to estimates of other functions. This fact has already been exploited in N-body scattering theory. Such calculus defines an algebra homomorphism. We now consider an intermediate topology $C_c^\infty(\mathbb{R}) \subset \mathcal{U} \subseteq C(\mathbb{R})$ such that $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators admits $\mathcal{U}$ functional calculus. Here $C_c^\infty(\mathbb{R})$ denotes the space of smooth functions of compact support. It is shown in

[27] that $\mathcal{U}$ is an algebra under pointwise multiplication.

Using this functional calculus and the abstract result from [28], we characterize the scalar type operators by showing that a densely defined linear operator H acting on a Hilbert space $\mathcal{H}$ is scalar type if it is of $(0,1)$ type $\mathbb{R}$ and $\| f(H) \| \leq \| f \|_\infty$ for all $f \in C_c^\infty(\mathbb{R})$.

1.7.3 The Semigroup theory of Operators

The theory of strongly continuous semigroup has been in existence for long, with some fundamental results of Hille Yosida[31] dating back to 1948. Generally in semigroup theory, one may focus entirely on the semigroup, and consider the generator as a derived concept or one may start with the generator and view the semigroup as the Laplace transform of the resolvent. More information on these can be found in [16].

Let X be a Banach space. A family $T = \{T(t) : 0 \leq t < \infty\}$ of linear operators from $X \to X$ is called a strongly continuous semigroup or C_o-semigroup if the following conditions are satisfied;

- $\| T(t) \| < \infty (ie \sup\{\| T(t)f \| : f \in X, \| f \| \leq 1\})$
- $T(t+s)f = T(t)T(s)f$ for all $f \in X$ and for all $t, s \geq 0$
- $T(0)f = f$ for all $f \in X$
- $t \to T(t)f$ is continuous for $t \geq 0$ and for each $f \in X$

T is called a C_o contraction semigroup if in addition to the conditions above

$$\| T(t)f \| \leq \| f \|$$

for all $t \geq 0$ and for all $f \in X$, i.e $\| T(t) \| \leq 1$ for each $t \geq 0$ and $\| f \| \leq 1$. Let T be a C_o semigroup on X. The (infinitesimal) generator H of $\{T(t)\}_{t\geq 0}$ is defined by

$$Hf = \lim_{t\to 0^+} \frac{T(t)f - f}{t} = \frac{d}{dt}T(t)f \mid_{t=0}$$

The domain of H being the set of all $f \in X$ for which the above limit exists. Consider a linear operator H acting on a Banach space X that generates a uniformly bounded holomorphic semigroup $\{e^{-\lambda H}\}_{Re(\lambda)\geq 0}$, where $Re(\lambda)$ denotes the real part of λ . It follows that in an equivalent norm; H, iH and $(-iH)$ generates a one parameter contraction group if and only if H is closed and densely defined and its spectrum contained in $[0, \infty)$, [9].

In this thesis therefore, we wish to characterize the scalar type operators by applying the semi-group theory of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators and the $\mathcal{U}$ functional calculus which can also be found in for example ([27],[28]). One of our major results would be to show that if a densely defined operator H on a Hilbert space $\mathcal{H}$ is a scalar type, then its dual H^* is also of scalar type and it admits a $\mathcal{U}$ functional calculus of scalar type.

1.7.4 Operators and Functionals

Definition 1.7.1

Let $\mathbb{F}$ be a field. A vector space over $\mathbb{F}$ is a non empty set denoted by V endowed with two operations called addition and scalar multiplication such that for all $x, y \in V$ and any scalar $\alpha \in \mathbb{F}$, we have that V is closed under addition and scalar multiplication.

Definition 1.7.2

A norm on the vector space V is a function

$$\| \, . \, \| : V \to \mathbb{R},$$

that satisfies the following axioms. For all $x, y \in V$ and $\alpha \in \mathbb{F}$,

1. $\| x \| = 0$ if and only if $x = 0$
2. $\| \alpha x \| = | \alpha | \| x \|$
3. $\| x + y \| \leq \| x \| + \| y \|$

$\| \, . \, \|$ is a seminorm if it satisfies the first two properties above.

Definition 1.7.3

A Banach space is a complete normed space $(V, \| \, . \, \|)$

Definition 1.7.4

An algebra is a vector space V over a field $\mathbb{F}$ endowed with the mapping $(x, y) \to xy$ of $V \times V \to V$ satisfying the following conditions for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$.

1. $x(yz) = (xy)z$
2. $x(y + z) = xy + xz$; $(y + z)\alpha = y\alpha + z\alpha$
3. $(\alpha x)y = x(\alpha y)$

Definition 1.7.5

Let $T : X \to Y$ be a mapping from a normed space X to a normed space Y. We say that T is a bounded operator if there exist a $c > 0$ such that $\| Tx \| \leq c \| x \|$, for all $x \in X$

Definition 1.7.6

An operator T is continuous at a point $x_o \in X$ if given $\epsilon > 0$, there exist a $\delta > 0$ such that $\| Tx - Tx_o \| < \epsilon$ whenever $\| x - x_o \| < \delta$ for all $x \in X$. In other words, T is continuous on X if it is continuous at every point of X.

Definition 1.7.7

Let X be a Hilbert space and $T : X \to X$ be a bounded linear operator on X into itself. Then for all $x, y \in X$, the adjoint operator T^* of T is defined as $\langle Tx, y \rangle = \langle x, T^* y \rangle$

We now state some properties of normal operators whose proofs can be found in [31].

1. $N(T) = N(T^*)$ where $N(T)$ denotes the nullspace of T.
2. $R(T)$ is dense in T if and only if T is injective. Here, $R(T)$ denotes the range of T
3. T is invertible if and only if there exist $c > 0$ such that $\| Tx \| \geq c \| x \|$ for every $x \in X$
4. If $Tx = \lambda x$ for all $x \in H$ and $\lambda \in \mathbb{C}$, then $T^* x = \overline{\lambda} x$.

Chapter 2

Literature Review

A fundamental problem in spectral theory involves finding a criteria for an operator to be of scalar type. The theory of scalar type spectral operators was initiated by Dunford[10]. His main interest was to generalize the theory of self adjoint operators defined on general Banach spaces. These are operators which admits an integral representation with respect to a countably additive spectral measure and hence has a functional calculus for bounded measurable functions on their spectrum. In 1960, Foias introduced a wider class of generalized spectral operators. Originally this class was defined with the help of spectral distributions instead of spectral measures. This class can be built up with real generalized spectral operators, that is, operators $H \in B(X)$ such that $\| e^{itH} \| = O(| t |^k)$ for some $k \geq 0$ and for real $k \to \infty$. This is equivalent to saying that there exist $C \geq 1$ and $k \geq 0$ such that $\| e^{itH} \| < C \mid 1+it \mid^k$ for every $t \in \mathbb{R}$ [26]. Other notable approaches come from, for example Kantorovitz [20], who established this using the boundedness of operators with real spectrum acting on a reflexive Banach space. Wermer [34] has shown that the scalar type operators in Hilbert space are those operators similar to normal operators. The similarity being implemented by a boundedly invertible self adjoint operator B such that $BH_iB^{-1} i = 1, ..., k$ are all normal. In [13], Foguel has shown that in any space, the sum (product) of two commuting spectral operators is a spectral operator if and only if the sum(product) of their scalar part is scalar type operator. This result has been extended by many authors for example, McCathy [4] considered operators acting on L^p spaces for $1 < p < \infty$. More recently, Gillepssie [15] obtained that the sum and product of two commuting scalar-type spectral operators on a weakly complete Banach lattice are scalar type spectral. A counterexample to this can be given on the von Neumann-Schatten classes C_p, for $1 < p \neq 2 < \infty$. One wonders under what conditions is the sum of two commuting scalar type spectral operators bounded. Well bounded operators have been introduced by Smart [32]. These operators have functional calculus for the absolutely continuous functions on some compact interval. Bade[2] has also given answers to the following questions. If τ is a family of commuting scalar type spectral operators, when are all the operators in the weakly closed algebra generated by τ also scalar type spectral, and what are operators in this weakly closed algebra generated by τ. Oleche [29] has defined the roots and exponentials of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators with $\sigma(H) \subseteq \mathbb{R}$ where H is a closed densely defined operator. We therefore wish to characterize the scalar type operators using the $\mathcal{U}$ functional calculus constructed in [28], and the semigroup theory of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators. Finally, we shall give applications in decomposability and some abstract Cauchy problems with appropriate boundary conditions.

2.1 Scalar type spectral operators

Scalar type spectral operator posses a larger functional calculus and give rise to spectral expansion. These operators can be represented as an integral with respect to the spectral measure. If $T \in B(X)$ is a scalar type spectral operator, then there exist a unique equicontinuous spectral measure P in $B(X)$ and an integrable function f such that $T := \int_{\Omega} f dP$. If X is a Banach space then this definition agrees with the well known definition due to N.Dunford [10].

2.2 Resolution of the Identity

Let Ψ be a σ algebra in a set Ω and $\mathcal{H}$ be a Hilbert space. A resolution of the identity on Ψ is a mapping

$$E : \Psi \to B(\mathcal{H})$$

satisfying the following properties for all $w, w', w'' \in \Psi$

1. $E(\emptyset) = 0, E(\Omega) = I$
2. Each $E(w)$ is a self adjoint projection
3. $E(w' \cap w'') = E(w')E(w'')$
4. If $w' \cap w'' = \emptyset$ then $E(w' \cup w'') = E(w') + E(w'')$

5. For every $x \in \mathcal{H}$ and $y \in \mathcal{H}$ the set function $E_{x,y}$ defined by $E_{x,y}(\omega) = (E(w)x, y)$ is a complex measure on Ψ.

If Ψ is the algebra of all Borel sets on a compact or locally compact Hausdorff space then $E_{x,y}$ is regular.

For the proof of the above properties, See [31].

The following are the consequences of the above properties;

Now since each $E(w)$ is a self adjoint projection, we have

$$E_{x,x}(\omega) = (E(w)x, x) = \| E(w)x \|^2$$

for all $x \in \mathcal{H}$. Property 4 implies that E is finitely additive. The main interest is whether E is countably additive, i.e whether the series $\sum_{n=1}^{\infty} E(w_n)$ converges in the norm topology of $B(\mathcal{H})$, to $E(w)$ where w is the union of the disjoint sets $w_n \in \Psi$. For a fixed $x \in \mathcal{H}$ and since $E(w_n)E(w_m) = 0$ when $n \neq m$ we have $E(w_n)x$ and $E(w_m)x$ are orthogonal to each other. From the property 5, we have

$$\sum_{n=1}^{\infty}(E(w_n)x, y) = (E(w)x, y)$$

for every $y \in H$. This implies that

$$\sum_{n=1}^{\infty}(E(w_n)x) = (E(w)x)$$

i.e the series converges in the norm topology of $\mathcal{H}$

2.3 Spectral Mapping Theorems

For a comprehensive theory on spectral mapping theorems we refer to [11].

Theorem 2.3.1 (Spectral mapping theorem for resolvents)
Let H be a closed operator on a Banach space X and $\lambda \in \rho(H)$. Then the following assertions hold:

1. $\sigma(R(\lambda, H)) \setminus \{0\} = (\lambda - \sigma(H))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(H)\right\}$

2. $\sigma_j(R(\lambda, H)) \setminus \{0\} = (\lambda - \sigma_j(H))^{-1}$ *for* $j = p, ap, su, r$

3. $r(R(\lambda, H)) = \frac{1}{dist(\lambda,\sigma(H))}$

4. *If H is unbounded, then* $0 \in \sigma(R(\lambda, H))$

Theorem 2.3.2 (Spectral mapping theorem for Semigroups)
Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup and let Γ be its infinitesimal generator. Then;

1. $\sigma(T(t)) \supset e^{t\sigma(\Gamma)}$

2. $\sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(\Gamma)}$

Moreover, if $\{T(t)\}_{t\geq 0}$ is strongly continuous semigroup of normal operators on a Hilbert space and Γ denotes its generator, then $\sigma(T(t)) = \overline{e^{t\sigma(\Gamma)}}$ where $\overline{e^{t\sigma(\Gamma)}}$ is the complex conjugate of $e^{t\sigma(\Gamma)}$.

2.4 Self adjoint Operators

Let $C_o(\mathbb{R})$ denote the space of bounded continuous functions on $\mathbb{R}$ which vanish as $x \to \infty$ and $\mathcal{H}$ denote a Hilbert space. Also let $\mathcal{B}(\mathcal{H})$ denote the space of bounded operators on the Hilbert space $\mathcal{H}$.

We now state the following theorem due to E.B Davies and which is a version of spectral theorem.

Theorem 2.4.1 ([7])

Let H be a self adjoint operator on the Hilbert space $\mathcal{H}$. There exist a unique linear map $f \to f(H)$ from $C_o(\mathbb{R})$ to $B(H)$ such that;

- *The map $f \to f(H)$ is an algebra homomorphism.*
- $\overline{f}(H) = f(H)^*$ *for all* $f \in C_o(\mathbb{R})$.
- $\| f(H) \| \leq \| f \|_\infty$ *for all* $f \in C_o(\mathbb{R})$
- *If* $w \notin \mathbb{R}$ *and* $r(w,s) := (w-s)^{-1}$ *then* $R(w,H) := (w-H)^{-1}$
- *If* $f \in C_o(\mathbb{R})$ *has support disjoint from* $\sigma(H)$ *then* $f(H) = 0$.

2.5 Almost analytic functional calculus and the semigroup theory of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators

In this section, we look at the main features of the functional calculi which can also be found in [28] and [3] and the semigroup theory of $(\alpha, \alpha+1)$

type $\mathbb{R}$ operators.

Definition 2.5.1

For $\beta \in \mathbb{R}$, we define C^β to be the space of smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $r \geq 0$ there exist $c_r > 0$ such that

$$\mid f^{(r)}(x) \mid := \mid \frac{d^r}{dx^r} f(x) \mid \leq c_r \langle x \rangle^{\beta - r}$$

for all $x \in \mathbb{R}$ and $\langle x \rangle = (1 + \mid x \mid^2)^{\frac{1}{2}}$

We now show that the space $\mathcal{U} := \cup_{\beta<0} C^\beta$ is an algebra under pointwise multiplication. We first show that $\mathcal{U}$ is linear.

Suppose $f, g \in \mathcal{U}$, then $f, g \in C_c^\infty(\mathbb{R})$, and there exist c_1 and c_2 in $(0, \infty)$ such that $\mid \frac{d^n}{dx^n} f(x) \mid \leq \frac{c_1}{\langle x \rangle^{n-\beta_1}}$ and $\mid \frac{d^n}{dx^n} g(x) \mid \leq \frac{c_2}{\langle x \rangle^{n-\beta_2}}$ for some $\beta_1, \beta_2 \in \mathbb{R}$.
Now

$$\begin{aligned}
\frac{d^n}{dx^n}\alpha(f+g)(x) \mid &= \mid \frac{d^n}{dx^n}(\alpha f(x) + \alpha g(x)) \mid \\
&= \mid \alpha \frac{d^n}{dx^n} f(x) + \alpha \frac{d^n}{dx^n} g(x) \mid \\
&\leq \mid \alpha \mid \frac{c_1}{\langle x \rangle^{n-\beta_1}} + \mid \alpha \mid \frac{c_2}{\langle x \rangle^{n-\beta_2}} \\
&\leq \frac{\mid \alpha \mid c_1 + \mid \alpha \mid c_2}{\langle x \rangle^{n-\beta}} \\
&= \mid \alpha \mid \frac{(c_1 + c_2)}{\langle x \rangle^{n-\beta}}
\end{aligned}$$

where $\beta = \max\{\beta_1, \beta_2\}$. This implies that $\mathcal{U}$ is linear.

Next, we show that if f and g are functions of class $\mathcal{U}$ then their product is also of class $\mathcal{U}$.

Let $f, g \in \mathcal{U}$ then f and g are n-times differentiable functions. It follows from Leibnitz rule that the nth derivative of the product is given by;

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^k g^{(n-k)}$$

$$\leq \sum_{k=1}^{\infty} D_k \langle x \rangle^{\beta_1 - k} \langle x \rangle^{\beta_2-(n-k)}$$

$$= c_n \langle x \rangle^{\beta_1+\beta_2-n}$$

for $c_n > 0$ and $D_k = \frac{n!}{k!(n-k)!}$. This implies that $fg \in \mathcal{U}$.

The algebra $\mathcal{U}$ contains the subalgebra $C_c^\infty(\mathbb{R})$ of all smooth functions with compact support.

The norm on $\mathcal{U}$ or $C_c^\infty(\mathbb{R})$ is therefore defined as

$$\| f \|_n := \sum_{r=0}^{n} \int_{-\infty}^{\infty} | f^{(r)}(x) | \langle x \rangle^{r-1} dx \, n \geq 1 \tag{2.1}$$

For $f \in \mathcal{U}$, let $\tilde{f}$ denote the following particular almost analytic extension of f to $\mathbb{C}$ of degree n defined by;

$$\tilde{f}(x+iy) := \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \tau \left(\frac{y}{\langle x \rangle} \right) \tag{2.2}$$

where τ is a $C_c^\infty(\mathbb{R})$ function is non negative such that $\tau(s) = 1$ if $| s |< 1$ and $\tau(s) = 0$ if $| s |> 2$.

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2}\left\{ \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right\} (\sigma_x + i\sigma_y) + \frac{\frac{1}{2} f^{n+1}(x)(iy)^n \sigma(x,y)}{n!} \}$$

and $\sigma(x,y) = \tau\left(\frac{y}{\langle x \rangle}\right)$ from which it follows that $\mid \frac{\partial}{\partial \bar{z}}\widetilde{f}(x,y) \mid = \mathcal{O}(\mid y \mid^n)$ as $\mid y \mid \to 0$ for each $x \in \mathbb{R}$. In particular $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for every $z \in \mathbb{R}$ and which is why we call $\tilde{f}$ an almost analytic extension of f. Note also that we can find $c' \in \mathbb{R}$ such that

$$\mid \frac{\partial}{\partial \bar{z}}\widetilde{f}(x,y) \mid \leq c'(\mid y \mid^n) \tag{2.3}$$

as $z \to x \in \mathbb{R}$.

If κ is a map such that $\kappa : \mathcal{U} \to B(X)$ then $f \to f(H)$ satisfy (1.5) and it is proved in [28], that for $n > \alpha \geq 0$

- $f(H)$ is norm convergent with $\| f(H) \| \leq C_\alpha \| f \|_{n+1}$ for some $C_\alpha > 0$ and doesn't depend on τ;
- the mapping extends to a bounded algebra homomorphism;
- if $f \in \mathcal{U}$ and $f = 0$ on a neighbourhood of $\sigma(H)$ then $f(H) = 0$;
- if $z \in i\mathbb{R}$ then $\frac{1}{z-H} \in \mathcal{U}$ and $f(\frac{1}{z-H}) = (z - H)^{-1}$

For an operator H of $(\alpha, \alpha+1)$-type $\mathbb{R}$, we associate each element $f \in \mathcal{U}$ with an operator $f(H) \in B(X)$ given by (1.5)

Theorem 2.5.2 ([29])
If H is a bounded operator with $\sigma(H) \subseteq \mathbb{R}$ and if $z \notin \mathbb{R}$ then;

$$(z - iH)^{-1} = \begin{cases} \int_0^\infty e^{-zt}e^{iHt}dt, & \text{if } Re(z) > 0; \\ -\int_0^\infty e^{-zt}e^{iHt}dt, & \text{if } Re(z) < 0. \end{cases}$$

Theorem 2.5.3 (Hille-Yosida Theorem)

Let X be a Banach space. Then H is a generator of a C_o- contraction semi-group if an only if H is closed, densely defined and for each $\lambda > 0$, $\lambda \in \rho(H)$ and $\| (\lambda - H)^{-1} \| \leq \lambda^{-1}$

PROOF. Let H be a generator of a C_o-contraction semi group, then $T(t) = e^{tH}$ is its semigroup and $T(t) \leq 1$. Now for each $\lambda > 0$ and $\lambda \in \rho(H)$, one has,

$$\begin{aligned} \| (\lambda - H)^{-1} \| &= \lim_{n\to\infty} \int_0^n e^{-\lambda t} T(t) dt \\ &\leq \lim_{n\to\infty} \int_0^n e^{-\lambda t} dt \\ &= \lim_{n\to\infty} \int_0^n \frac{1}{e^{\lambda t}} dt = \lambda^{-1} \end{aligned}$$

Next, we show that the generator H belongs to $B(X)$, i.e $\| T(t) - I \| \to 0$ as $t \to 0$.

Now

$$\begin{aligned} \| T(t) - I \| &= \| \sum_{n=0}^{\infty} \frac{t^n H^n}{n!} - I \| \\ &\leq \sum_{n=1}^{\infty} \frac{t^n \| H \|^n}{n!} \\ &= e^{t\|H\|} - 1 \to 0 \ as \ t \to 0 \end{aligned}$$

□

Corollary 2.5.4 ([29])

If iH is a generator of a group of isometries $\{T(t)\}_{t\geq 0}$, then for all $\lambda \in \mathbb{C}$, with $R(\lambda) \neq 0$, $\lambda \in \rho(iH)$;

$$(\lambda - iH)^{-1} = \begin{cases} \int_0^\infty T(t)e^{-\lambda t}dt, & \text{if } Re(\lambda) > 0; \\ -\int_0^\infty T(t)e^{-\lambda t}dt, & \text{if } Re(\lambda) < 0. \end{cases} \tag{2.4}$$

Theorem 2.5.5 ([26])

Let H be a densely defined linear operator on a Banach space X, then the following two statement are equivalent.

(i) H generates integrated semigroup.

(ii) There exist real constants w and C and $k \in \mathbb{N}_o$ such that $| R(\lambda, H) | \leq C(1+ | u |)^k$ for all $u \in \mathbb{C}$ with $Re(\lambda) > w$.

Theorem 2.5.6 ([27])

If $f : [0, \infty) \to \mathbb{C}$ is such that

$$| \frac{d^r}{dx^r} f(x) | \leq c_r \langle x \rangle^{\beta - r}$$

for some $\beta < 0$ and for all $r \geq 0$ and for all $x \geq 0$ and H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$ with $\sigma(H) \subseteq [0, \infty)$, then $f(H)$ is uniquely determined and

$$\| f(H) \| \leq \| f \|_{n+1}$$

for $k > 0$ whenever $n > \alpha$

Theorem 2.5.7 ([28])

Let H be an operator of $(\alpha, \alpha+1)$ type $\mathbb{R}$ for some $\alpha \geq 0$. If $w \in \mathbb{C} \backslash \mathbb{R}$ and $r_w(x) := (w - x)^{-1}$ for all $x \in \mathbb{R}$, then $r_w \in \mathcal{U}$ and $r_w(H) = (w - H)^{-1}$.

Theorem 2.5.8 ([28])

If $f \in \mathcal{U}$ and H is self adjoint on a Hilbert space $\mathcal{H}$, then

$$\| f(H) \| \leq \| f \|_\infty .$$

Definition 2.5.9

If $f \in \mathcal{U}$ and H is self adjoint, on a Hilbert space $\mathcal{H}$, then the functional calculus

$$k : \mathcal{U} \ni f \to f(H) \in B(H)$$

can be extended to a unique map

$$\tilde{k} : C_o(\mathbb{R}) \ni f \to f(H) \in B(H)$$

such that;

1. $f \to f(H)$ is an algebra homomorphism.
2. $\bar{f}(H) = f(H)^*$
3. $\| f(H) \| \leq \| f \|_\infty$
4. If $w \in \mathbb{C} \setminus \mathbb{R}$ and $r_w := (w - s)^{-1}$ then $r_w(H) = (w - H)^{-1}$

Moreover the functional calculus is unique subject to these conditions. Note that for information on the above you can See[28]

Theorem 2.5.10 ([28])

If H is of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ for some $\alpha > 0$, then H admits $C_c^\infty(\mathbb{R})$ functional calculus.

Definition 2.5.11

Let $H \in B(X)$, then the semigroup e^{tH} is said to be bounded if there exist a constant $C \geq 1$ and $\gamma \geq 0$ such that

$$\| e^{tH} \| \leq Ce^{t\gamma} \tag{2.5}$$

for all $t \geq 0$.

Theorem 2.5.12 ([29])

Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$, and $\| e^{iHt} \| \leq C(1+ | t |)^{\alpha}$ where α is a non negative integer. Then H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$

REMARK 2.5.13

The above theorem is consistent with theorem 2.4.5 hence one can deduce that if Theorem 2.4.12 holds then H generates an integrated semigroup.

The following are two immediate consequences arising from the above theorem;

Corollary 2.5.14

If $\alpha \geq 0$ is the minimal constant such that inequality in Theorem 2.4.12 holds, then $T(t) = e^{iHt}$ is α contractive

Corollary 2.5.15

If $\{T(t)\}_{t\geq 0}$ is 0 contractive and $C = 1$, then H is of $(0,1)$-type $\mathbb{R}$ and the inequality in Theorem 2.4.12 reduces to a contraction semi-group and $(0, \infty) \subseteq \rho(H)$

We now state another version of Hille Yosida theorem given below;

Theorem 2.5.16 (Hille Yosida[31])

A closed densely defined linear operator H on a Banach space X is the infinitesimal generator of a semigroup $T(t)$ if and only if there exist a constant C and γ such that for every $\lambda > \gamma$, $(\lambda I - H)$ is invertible with

$$\| (\lambda I - H)^{-m} \| \leq C(\lambda - \gamma)^{-m} \tag{2.6}$$

$\forall m \in \mathbb{N}$

The following is an immediate consequence of the above theorem.

Corollary 2.5.17

If $m = 1$ (2.6) reduces to

$$\| (\lambda I - H)^{-1} \| \leq C(\lambda - \gamma)^{-1} \tag{2.7}$$

Theorem 2.5.18 (Stones theorem[31])

Every one parameter group of unitary transformation is of the form e^{iHt} with H self adjoint.

Definition 2.5.19

If γ is a rectifiable Jordan curve and $n(\gamma : c)$ denotes the winding number for $c \in \mathbb{C} \setminus \gamma$ and $n(\gamma : c) = 1$, then γ is said to be positively oriented. A positive Jordan system is a collection $\Gamma := \{\gamma_1, ..., \gamma_m\}$ of pairwise disjoint rectifiable Jordan curves such that for all $c \notin \gamma_i$, for all i,

$$n(\Gamma : c) := \sum_{i=1}^{m} n(\gamma_i : c) = 0 \, or \, 1$$

The inside(ins) of Γ is the set $ins\Gamma := \{c \in \mathbb{C} : n(\Gamma : c) = 1\}$

Theorem 2.5.20 (Green's theorem)

If Γ *is a smooth positive Jordan system with* $G := ins\Gamma$, $f \in C(\overline{G}) \cap C^1(G)$ *and* $\frac{\partial f}{\partial \bar{z}}$ *is integrable over* G *then*

$$\int_\Gamma f(z)dz = 2i \int_G \frac{\partial f}{\partial \bar{z}} dxdy \tag{2.8}$$

Corollary 2.5.21 ([21])

Let H *be an operator on* X *with* $G \subset \rho(H)$ *and*
$g(z) := f(z)(z-H)^{-1}$ *is such that* $g \in C(\overline{G}) \cap C^1(G)$ *and* $\frac{\partial g}{\partial \bar{z}}$ *is integrable over* G *then*

$$\int_G \frac{\partial}{\partial \bar{z}} f(z)(z-H)^{-1} dxdy = \frac{1}{2i} \int_\Gamma f(z)(z-H)^{-1} dz \tag{2.9}$$

2.6 Distributive derivatives

Let $C_c^\infty(\Omega)$ denote the space of smooth functions with compact support contained in Ω. One can define a distribution to be a continuous linear functional $\phi : C_c^\infty(\Omega) \to \mathbb{C}$ where the underlying space is given the usual topology. If $f \in Loc^1(\Omega)$ then f may be identified with a distribution ϕ_f given by the formula

$$\phi_f(g) = \int_\Omega f(x)g(x)d^N x$$

Given a multiindex α, the week derivative $D^\alpha \phi$ of the distribution ϕ is given by $(D^\alpha \phi)(g) = (-1)^{|\alpha|} \phi(D^\alpha g)$ This is in line with the usual notation of differentiation if we associate ϕ with differentiable function on Ω and therefore allows us to extend differentiation to a wider class of functions.

2.7 Asymptotic analysis and Notations

A symptotic analysis is a method of classifying limiting behavior by concentrating on some trend. The notations provide a convenient language for handling statements pertaining to the order of growth. Asymptotic expansion of the function $f(x)$ is an expression of that function in terms of infinite series, the partial sum of which do not necessarily converge but such that taking any initial sum provides an asymptotic function for f. The successive terms provide a more and more accurate description of the order of the growth. Let $f(x)$ and $g(x)$ be two functions defined on some subset of real numbers. We say $f(x)$ is $\mathcal{O}(g(x))$ as $x \to \infty$ if and only if there exist numbers x_o and M such that $| f(x) |\leq Mg(x)$ for $| x - x_o |< \delta$. Note, in mathematics, both asymptotic behaviours near ∞ and near a constant a are considered. In computational complexity, only asymptotic near infinity are considered.

2.8 Basic L^p Theory

Let Ω be lebesque measurable subset of $\mathbb{R}^N$. For $p \in [0, \infty)$, let $L^p(\Omega)$ denote the space of measurable functions $f : \Omega \to \mathbb{C}$ which have finite norm denoted by

$$\| f \|= \left(\int_{\Omega} | f(x) |^p \, d^N x \right)^{\frac{1}{p}}$$

if $p \in [1, \infty)$ otherwise, $\| f \|_{\infty}= min\{\lambda : meas\{x :| f(x) |> \lambda\} = 0$. Identifying functions which are equal excepts on the set of measure zero, the $L^p(\Omega)$ becomes a Banach space. In particular $L^2(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x)d^N x$$

Let $p, q, r \in [1, \infty)$ and let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then from Holders inequality, we have that if

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

then

$$\| fg \| \leq \| f \|_p \| g \|_q$$

From Young's inequality, we have that if

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

then for functions $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, the convolution $f * g$ is defined as follows

$$(f * g)(y) = \int_{\mathbb{R}^N} f(y - x)g(x)d^N x$$

and $\| f * g \|_r \leq \| f \| \| g \|_q$. Let $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ then the Fourier transform of f denoted by $\hat{f}$ is defined by

$$\hat{f}(y) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(x)e^{-ix.y} d^N x$$

for $1 \leq p \leq 2$, and if p is conjugate to q, we've

$$\| \hat{f} \|_q \leq (2\pi)^{N/2-N/P} \| f \|_p$$

for all such p. This means that the map $F : f \to \hat{f}$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ for all such p.

2.9 Application to Decomposability and Cauchy Problems

If H is a bounded operator on a complex Banach space X, then H is decomposable if whenever $\{U_1, U_2, ..., U_n\}$ forms an open cover of X, there exists a closed T- invariant subspaces Y_k such that X can be expressed as $X = Y_1 + Y_2 + ..., +Y_n$ and $\sigma(H \mid Y_k) \subseteq U_k$ for $k = 1, 2....,$. Since this class comprises of normal operators on a Hilbert space, it follows that the class is of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators. In our study, we shall show that if H, is of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators and that it generates a strongly continuous group on a Banach space, then its resolvent is decomposable whenever $\lambda \in \rho(H, X)$.

For application to Cauchy problems, we consider the general abstract Cauchy equation given by;

$$\begin{cases} \text{u'(t)=-Hu(t)}, & t \geq 0; \\ \text{u(0)=x}, & x \in X. \end{cases} \tag{2.10}$$

It is well known that a function $u(.) : [0, \infty) \to D(H)$ with $u(.) \in ([0, \infty); X)$ and $u(0) = x$ which satisfies (2.10) is a solution of (2.10).

In studying (2.10), the notion of integrated semigroups comes in handy. This is a class which comprises of the one parameter semi-group and the cosine families. However, it is also important to note that some classes of abstract Cauchy equations exist where the elements of e^{-tH} is not bounded operators, for example the Schrodinger operators acting on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $p \neq 2$. To deal with such problems, one needs to find larger sets of functions f giving rise to bounded operators in form of $e^{-tH}f(H)$ such that the solution of (2.10) exist. In [18] it had been realized that (1.5) can be used to study Schrodinger operators on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $p \neq 2$, in which case the general solution $u(t) = e^{-itH}$ of the Schrodinger equation is unbounded. This means that in (2.10) one must look for a suitable functional calculus involving H and e^{-tH}, and so the notion of $\mathcal{U}$ functional calculus comes in handy. In our study therefore, we shall apply the $\mathcal{U}$ functional calculus for $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator H satisfying (1.5) to study some abstract Cauchy equations of the form given by (2.10) where the solution denoted by $u(t)$ is unbounded.

Chapter 3

Scalar type operators

3.1 Basic properties of scalar type operators

In this section, we highlight some basic properties of the scalar type operators relevant to our study area. In particular, we emphasize on the basic properties of the semigroups whose generators are scalar type operators and which forms an integral part in the characterization of scalar type operators. Let X be a Banach space and X^* be its dual. An operator on X will be a linear operator which is not necessarily bounded and whose domain and range are subset of X. We now state and prove some of the properties of scalar type operators.

Our first result relates self adjoint operators and $(0,1)$ type $\mathbb{R}$ operators.

Theorem 3.1.1 ([29])

Let H be a self adjoint operator on a Hilbert space $\mathcal{H}$, then H is of $(0,1)$ type $\mathbb{R}$

Our next three theorems give a characterization of a strongly continuous positive scalar type operators acting on a Hilbert space $\mathcal{H}$. The proofs of the theorems can be found in [33].

Theorem 3.1.2

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of positive scalar type operators on $\mathcal{H}$. Then the infinitesimal generator H, is scalar type with spectrum contained in some interval $(-\infty, c]$, and there exist a resolution of identity $F(.)$ defined on the Borel set such that $T(t) = \int_{-\infty}^{c} e^{\lambda t} dF(\lambda)$ where $F(.)$ is the resolution of the identity for H.

Theorem 3.1.3

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of positive scalar type operators on a Hilbert space $\mathcal{H}$, then $T(t)$ is self adjoint.

Theorem 3.1.4

Let $\{T(t)\}_{t\geq 0}$ be a strongly uniformly continuous semigroup of positive scalar type operators on a Hilbert space $\mathcal{H}$, then the infinitesimal generator H is a scalar type operator.

The following is a consequence of Theorem 3.1.3 and Theorem 3.1.4

Corollary 3.1.5

If H satisfy (1.4) and $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of positive scalar type operators on a Hilbert space $\mathcal{H}$, then $T(t)$ is self adjoint and H is of $(0, 1)$ type $\mathbb{R}$ and hence a scalar type operator.

PROOF. The first part of the Corollary follows from Theorems 3.1.4 and 2.4.18, and the last part follows from Theorem 3.1.1. $\square$

3.2 Pseudo Hermitian and scalar type operators

In [14], Foias proved that if $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of scalar type operators on a Hilbert space $\mathcal{H}$, with a uniformly bounded resolution of identity, then $T(t)$ is similar to a semigroup of normal operators. That is a normal operator on a bounded Banach space is normal, if it is a scalar type operator with all the projections in the range of its resolution of the identity hermitian.
We now give the characterization of a bounded spectral operator called Pseudo-hermitian (p.h) operators. These are scalar type spectral operators with real spectrum. This terminology is motivated by the fact that a pseudo-hermitian operator S on Hilbert space is similar to a hermitian operator. This is due to Mackey's [24] result on the existence of a non singular operator P such that the resolution of the identity $E(.)$ of H satisfies $E(\delta) = PE(\delta)P^{-1}$ where δ are Borel sets defined on the complex plane, and $E(\delta)$ is self adjoint. Thus the Pseudo-hermitian operators play the same role in Banach space as hermitian operators play in a Hilbert space. If H is a spectral scalar type operator, with a resolution of identity, then H can be defined as $H := A + iB$ where A, B are pseudo-hermitian operators. The operators A and B commute and any operator commuting with H, must also commute with A and B.

We now state and prove the sufficient condition for a bounded operator to be a pseudo hermitian operator.

Theorem 3.2.1

Let H be a bounded linear operator on a Hilbert space $\mathcal{H}$, then H is *pseudo-hermitian operator, if and only if the group* $\{e^{-2\pi i\lambda H}\}$ *is uniformly bounded for all* $\lambda \in \mathbb{R}$

PROOF. We only need to prove the sufficient condition. If $\{e^{-2\pi i\lambda H}\}$ is uniformly bounded, then there exist $M > 0$ such that $\| e^{-2\pi i\lambda H} \| \leq M$. Since H is Hermitian, it follows from [24] that there exist of a non singular operator P such that $P^{-1}e^{i\lambda H}P$ for $\lambda \in \mathbb{R}$ forms a group of unitary operators. This group generates a self conjugate operator S since iS generates a strongly continuous group of isometries. It follows from Stones theorem and Theorem 3.1.1 that S is of $(0,1)$ type $\mathbb{R}$ hence a scalar type operator. Now $S = P^{-1}HP$ and hence it is bounded. Hence S is hermitian and $H = PSP^{-1}$ implies that H is Hermitian, ie Pseudo-hermitian operator and it is also of $(0,1)$ type $\mathbb{R}$. □

The following is an immediate consequence of Theorem 2.4.12;

Corollary 3.2.2

If $\alpha = 0$, *then* H *is of* $(0,1)$ *type* $\mathbb{R}$ *and* $\| e^{iHt} \| \leq C < \infty$. *In particular,* H *is a pseudo hermitian operator, and so it is a scalar type operator.*

PROOF. The proof follows from the fact that $(0,1)$ type $\mathbb{R}$ operators are self adjoint operators and so they generate a semi-group which is also self adjoint, and therefore by Stones Theorem H is self adjoint and therefore it is a pseudo-hermitian operator, hence a scalar type operator. □

We now state and prove a theorem relating the pseudo Hermitian operator and the space $L^1(\mathbb{R})$.

Theorem 3.2.3

If H is a bounded linear operator on a Hilbert space, then the following two statements are equivalent.

(i) H is a Pseudo-hermitian operator.

(ii) For every $f \in L^1(\mathbb{R})$,

$$\| \int_{\mathbb{R}} f(\lambda) e^{-2\pi i \lambda H} d\lambda \| \leq M \| \hat{f} \|,$$

where the norm on the left is of the operator norm, $\hat{f}$ is the Fourier transform of f and $\| \, . \, \|$ is the supnorm.

PROOF. (i) implies (ii) follows from the definition of Pseudo-hermitian operator H for $f \in L^1(\mathbb{R})$ and so we only need to prove that (ii) implies (i). Now since (ii) is uniformly bounded, it implies that $\| e^{-2\pi i \lambda H} \| \leq M < \infty$ by Corollary 3.2.2. Also H hermitian, and so it follows from [24], that $P^{-1} e^{i\lambda H} P$ forms a group of unitary operators where P is non singular. It now follows from Stones Theorem that H is self adjoint and by Theorem 3.1.1 that H is $(0, 1)$ type $\mathbb{R}$ operator. Finally invoking Corollary 3.2.2 gives the required result. □

Chapter 4

Characterization of scalar type operators

4.1 Introduction

In this chapter, we give the characterization of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator H acting on a Hilbert space $\mathcal{H}$ for the case when $\alpha = 0$. This is given comprehensively in the next section.

4.2 Characterization of scalar type operators of (0,1)-type $\mathbb{R}$

Our first result relates the $(0,1)$ type $\mathbb{R}$ operator and its generator.

Theorem 4.2.1

H is of $(0,1)$-type $\mathbb{R}$ with the constant $C = 1$ in equation (1.4) if and only if iH is a generator of a one parameter group of isometries on X.

PROOF. Suppose H is of $(0,1)$-type $\mathbb{R}$ with $C = 1$, then it follows from Corollary 3.2.2 that H is a Pseudo-Hermitian operator, and hence a scalar type operator. Also the resolvent of H is bounded and its semigroup $T(t) = e^{itH}$ is also uniformly bounded by Theorem 2.4.12. Also from Theorem 3.1.4, the semigroup is self adjoint, hence it follows by Theorem 2.4.18 that H is self adjoint. Further, applying Theorem 3.1.5 implies that H, is indeed a scalar type operator. Now, iH also generates Laplace transform defined by

$$(\lambda - iH)^{-1} = \int_0^\infty T(t)e^{-\lambda t}dt$$

where $T(t) = e^{itH}$ for all $\lambda \in \mathbb{C}$ with $\lambda \in \rho(H)$. Since iH generates a group of isometries, it follows that H is densely defined. Now for $\lambda > 0$; $(\lambda - iH)^{-1}f$ is the Laplace transform of $T(t)f = e^{itH}f$ given for f in the domain of H which is a bounded operator. Conversely suppose H is densely defined and Theorem 2.4.12 holds, then $T(t)$ is a semigroup. From the uniform boundedness principle, $T(t)$ is uniformly bounded on compact intervals. From Corollary 2.4.4 and density of $D(H)$, implies that $T(t)$ is strongly continuous. It follows that iH is a generator of one parameter group $T(t)$ and this completes our proof. □

The next result relates $(0,1)$-type $\mathbb{R}$ operators with scalar type operators.

Theorem 4.2.2

A densely defined linear operator H acting on a Hilbert space $\mathcal{H}$, is scalar type if it is of $(0,1)$ -type $\mathbb{R}$ and $\| f(H) \| \leq \| f \|_\infty$ for each f in the algebra of smooth functions $\mathcal{U}$

PROOF. Let H be an operator acting on Hilbert space $\mathcal{H}$ and $\sigma(H) \subseteq \mathbb{R}$ then H is a Pseudo Hermitian operator and it follows from Corollary 3.2.2 that it is a scalar type operator. Since iH generates a one parameter group which is of scalar type, and H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator, then H admits a functional calculus given by (1.5) for all $f \in \mathcal{U}$. Now, (1.5) is continuous by (1.4) and (2.3). From Theorem 2.4.8 we see that the functional calculus (1.5) converges absolutely. Since H is Hermitian, it follows from Riesz Representation Theorem that for $f \in \mathcal{U}$ there exist a complex Borel measure μ on $\sigma(H)$ such that

$$f(H) = \int_{\sigma(H)} f(z)\mu dz.$$

This implies that H is a scalar type operator, hence the proof. □

The next result gives the relationship between a closed densely defined operator H and iH,

Theorem 4.2.3

Let H be a closed densely defined operator on a Hilbert space $\mathcal{H}$. Then iH is a scalar type operator if H is also a scalar type operator.

PROOF. Let H be a closed densely defined operator on a Hilbert space $\mathcal{H}$ satisfying (1.4), and $\sigma(H) \subseteq \mathbb{R}$. It follows from Theorem 2.4.16 that $\| (\lambda I - iH)^{-m} \| \leq C(\lambda - \gamma)^{-m}$ for all $m \in \mathbb{N}$ and $C > 0$. Also from Laplace transform and for $\lambda > 0$ and $x \in X$ one has;

$$(\lambda I - iH)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)xdt \tag{4.1}$$

Using standard properties of Laplace transform and semigroup Theory we have, By integration by parts

$$\begin{aligned}
\| R_\lambda^m x \| &= \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t} \| T(t)x \| \, dt \, where \, m = 1,2,3... \\
&= \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t} \| e^{iHt}x \| \, dt \\
&\leq \| x \| \frac{C}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t}(1+ | t |)^\alpha dt \\
&\leq \frac{C}{(m-1)!}\int_0^\infty t^{m-1}e^{(\alpha-\lambda)t}(1+ | t |)^\alpha dt \; with \; \| x \| = 1
\end{aligned}$$

Applying corollary (2.4.15) and taking $m = 1$, we have;

$$\| R_\lambda \| \leq \int_0^\infty e^{-\lambda t} dt \leq \frac{1}{\lambda}$$

This implies that iH generates a contraction semi-group given by $T(t) = e^{iHt}$. Since $T(t)$ is self adjoint, it follows that the resolvent set $(\lambda I - iH)^{-1}$ is also self adjoint and so $(\lambda I - iH)$ is also self adjoint. If $(\lambda I - iH)$ is self adjoint then H is also self adjoint and so it is of $(0,1)$-type $\mathbb{R}$ and hence by Corollary 3.2.2 it is a scalar type operator. Now since e^{iHt} is a scalar type operator with bounded resolution of the identity, then it follows from spectral theorem that a unique projection valued measure $E(.)$ from the Borel σ-field on $\mathbb{R}$ exist such that the generator iH of $T(t)$ is also a scalar type operator with the same bounded resolution of the identity E such that

$$iH = \int_{\mathbb{C}} (i\lambda)E(d\lambda)$$

and this completes our proof. □

The next theorem relates an operator H and its adjoint H^* on the Hilbert space.

Theorem 4.2.4

If $\mathcal{H}$ is a Hilbert space and H is a bounded scalar type operator on $\mathcal{H}$, then H^* is also a scalar type operator and it admits a $\mathcal{U}$ functional calculus.

PROOF. Let H be a scalar type operator and $\mathcal{U}$ denote algebra of smooth functions. Also let $C_c^\infty(\mathbb{R})$ be the subalgebra of $\mathcal{U}$ generated by $\{g_\mu : g_\mu(\lambda) = (\mu - \lambda)^{-1}, \mu \notin \mathbb{R}\}$. It follows from theorem 2.4.7 that $g_\mu(\lambda) \in \mathcal{U}$ and $g_\mu(H) = (\mu - H)^{-1}$.

For $f \in C_c^\infty(\mathbb{R})$, and for all $z \notin \mathbb{R}$, the $\mathcal{U}$ functional calculus (1.5) yields;

$$f(H) := -\frac{1}{\pi}\int_G \frac{\partial}{\partial \bar{z}}\tilde{f}(z)(z-H)^{-1}dxdy = -\frac{1}{2\pi i}\int_\Gamma \tilde{f}(z)(z-H)^{-1}dz.$$

It follows from Theorem 2.4.8 that $\| f(H) \|$ is bounded for each $f \in C_c^\infty(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{U}$, hence the homomorphism $f \to f(H)$ extends to a continuous homomorphism $h : C_c^\infty(\mathbb{R}) \to B(X)$.

It follows that $h(g_\mu) = (\mu - H)^{-1}$ for $\mu \notin \mathbb{R}$ and its dual

$$h(g_\mu)^* = [(\mu - H)^{-1}]^* = [(\bar{\mu} - H^*)]^{-1} \tag{4.2}$$

Now;

$$\begin{aligned}
f(H) : &= -\frac{1}{\pi}\int_G \frac{\partial}{\partial \bar{z}}\tilde{f}(z)(z-H)^{-1}dxdy \\
&= -\frac{1}{2\pi i}\int_\Gamma f(z)[(z-H)^{-1}]dz \\
&= -\frac{1}{2\pi i}\int_\Gamma f(z)[(z-H)^{-1}]^*dz \\
&= -\frac{1}{2\pi i}\int_\Gamma f(z)[(z-H)^*]^{-1}dz
\end{aligned}$$

and so

$$(f(H))^* = -\frac{1}{2\pi i}\int_\Gamma f(z)[(z-H)^*]^{-1}dz \quad \forall f \in \mathcal{U}$$

Since $D(H)$ is dense. By [11], there exist a spectral measure G of class X^* defined on the Borel sets with values in $L(X^*)$ such that

$$(f(H))^* = \int f(\lambda)G(d\lambda) \quad \forall f \in \mathcal{U}$$

Hence H^* is a scalar type operator and this completes our proof. $\square$

Chapter 5

Some application of scalar type operators

5.1 Application to Decomposability

Definition 5.1.1 (Decomposable Operator)

A bounded operator H on a complex Banach space X is decomposable provided that whenever $\{U_1, U_2, ...U_n\}$ is an open cover of $\mathbb{C}$, there exists closed, H-invariant subspaces Y_k such that $X = Y_1, +Y_2 + ..., Y_n$ and $\sigma(H \mid Y_k) \subseteq U_k$, $k = 1, 2, ..., n$.

This class of operators contains all normal operators on a Hilbert space and compact Banach space operators hence they are of $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators. We now state the following Theorem due to Albrecht and Eschmier [1] and which gives the necessary and sufficient condition for a bounded operator $H \in B(X)$ to be decomposable.

Theorem 5.1.2 ([1])

A bounded operator $H \in B(X)$ is decomposable if and only if H has

Bishop's property (β) and the decomposition property (δ).

We now define these properties.

Definition 5.1.3 (Bishop's Property (β))

Let X be a Banach space and Ω an open subset of the plane. Let $Hol(\Omega, X)$ denote the space of analytic functions from Ω to X. Then $Hol(\Omega, X)$ is a Fretchet space with respect to uniform convergence on the compact subsets of Ω. The operator $H \in B(X)$ is said to possess Bishop's property (β), provided that for every open subset $\Omega \subset \mathbb{C}$, $H_\Omega : Hol(\Omega, X) \to Hol(\Omega, X)$, $H_\Omega f(z) = (z - H)f(z)$ is injective with closed range.

Definition 5.1.4 (Decomposition Property (δ))

If F is a closed subspace of $\mathbb{C}$, then the glocal analytic spectral subspace $X_H(F)$ is $X_H(F) = X \cap ranH_{\mathbb{C}\setminus F}$, that is $x \in X_H(F)$ if there exist an analytic function $f : \mathbb{C}\setminus F \to X$ so that $(H-\lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\setminus F$
A bounded linear operator $H \in B(X)$ has the decomposition property (δ) if $X = X_H(\overline{U}) + X_H(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$.
Albrecht and Escheneier [1] established the remarkable fact that the properties (β) and (δ) are dual to each other. Indeed, $H \in B(X)$ has property (β) (resp (δ)) if and only if H^* has (δ) (resp.(β)).

We shall greatly use the following formulation by Laursen and Neumann [23]

Theorem 5.1.5 ([23])

Let $H \in B(X)$ and D be a closed disk that contains $\sigma(H)$, and let V be an open neighborhood of D. Suppose that there exist a totally disconnected *compact subset E of the boundary of D, a locally bounded function $\omega :$ $V \setminus E \to (0, \infty)$ and an increasing function $\gamma : (0, \infty) \to (0, \infty)$ such that log of γ has an integrable singularity at zero and $\gamma(dist(\lambda, \partial D)) \parallel x \parallel \leq$ $\omega(\lambda) \parallel (H - \lambda)x \parallel$ for all $x \in X$ and $\lambda \in V \setminus \partial D$, then H has property (β)*

In particular, the Theorem provide sufficient conditions in terms of the norms of resolvents for bishop's property (β).

Lemma 5.1.6

Let H be generator of arbitrarily continuous semigroup on a Banach space X and let $\lambda, \mu \in \rho(H)$, then $R(\lambda, H)R(\mu, H) = R(\mu, H)R(\lambda, H)$

PROOF. The proof follows immediately from the well known resolvent identity;
$R(\lambda, H) - R(\mu, H) = -(\mu - \lambda)R(\lambda, H)R(\mu, H)$ for all $\lambda, \mu \in \rho(H)$. □

Lemma 5.1.7

Let H be as in Lemma 5.1.6 and let $T = R(\lambda, H)$. Then $\mu \in \rho(T)$ if and only if $\lambda - \frac{1}{\mu} \in \rho(T)$. In this case, we have

$$(\mu - T)^{-1} = \frac{1}{\mu}I + \frac{1}{u^2}R(\lambda - \frac{1}{\mu}, H) \tag{5.1}$$

PROOF. From equation (5.1), $R(\lambda - \frac{1}{\mu}, H) - T = [\lambda - (\lambda - \frac{1}{\mu})]TR(\lambda - \frac{1}{\mu}, H)$ which implies $\mu T = (\mu - T)R(\lambda - \frac{1}{\mu}, H) = R(\lambda - \frac{1}{\mu}, H)(\mu - T)$. Multiplying by $(\lambda - H)$ and dividing through by μ yields

$$I = \frac{1}{\mu}(\lambda - \frac{1}{\mu} - H + \frac{1}{\mu})R(\lambda - \frac{1}{\mu}, H)(\mu - T)$$

$$= \frac{1}{\mu}(I + \frac{1}{\mu}R(\lambda - \frac{1}{\mu}, H))(\mu - T)$$

Thus

$$(\mu - T)^{-1} = \frac{1}{\mu}I + \frac{1}{\mu^2}R(\lambda - \frac{1}{\mu}, H).$$

□

The next theorem which is a major result in this section indicates that the kind of resolvent we are dealing with here are decomposable.

Theorem 5.1.8

If H is a generator of arbitrarily strongly continuous semigroup on Banach Space X with $\sigma(H, X) \subset \{z : Re(z) \leq c\}$ on a Banach space X, then the resolvent operator $R(\lambda, H)$ is decomposable for all $\lambda \in \rho(H, X)$.

Proof. Let H be the generator of strongly continuous semigroup with $\sigma(H, X) \subset \{z : Re(z) \leq c\}$ on a Banach space X. Let $\lambda, \mu \in \rho(H)$ and $T = R(\lambda, H)$. By the Hille Yosida Theorem 2.4.16, we have

$$\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{Re(\lambda - \frac{1}{\mu}) - c}$$

where $M > 0$ is a constant.

Now, by the spectral mapping theorem, we get

$$\sigma(T) \;=\; \sigma(R(\lambda, H)) = \{\frac{1}{\lambda - it} : t \in \mathbb{R}\} \cup \{0\}$$

Let

$$\begin{aligned} \omega &= \frac{1}{\lambda - it} = \frac{1}{Re(\lambda) + i(Im(\lambda) - t)} \\ &= \frac{Re(\lambda) - i(Im(\lambda) - t)}{(R(\lambda))^2 + (Im(\lambda) - t)^2} \end{aligned}$$

where $\lambda = Re(\lambda) + iIm(\lambda)$

Now

$$\begin{aligned} \omega - \frac{1}{2Re(\lambda)} &= \frac{Re(\lambda) - i(Im(\lambda) - t)}{(R(\lambda))^2 + (Im(\lambda) - t)^2} - \frac{1}{2Re(\lambda)} \\ &= \frac{2Re(\lambda)(Re(\lambda) - i(Im(\lambda) - t)) - (Re(\lambda))^2 - (Im(\lambda) - t)^2}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - t)^2)} \end{aligned}$$

Therefore

$$\begin{aligned} \left|\omega - \frac{1}{2Re(\lambda)}\right|^2 &= \left|\frac{(Re(\lambda))^2 - (Im(\lambda) - t)^2 - i2Re(\lambda)(Im(\lambda) - t)}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - t)^2)}\right|^2 \\ &= \frac{((Re(\lambda))^2 - (Im(\lambda) - t)^2)^2 + 4(Re(\lambda))^2(Im(\lambda) - t)^2}{4(Re(\lambda))^2((Re(\lambda))^2 + (Im(\lambda) - t)^2)^2} \\ &= \frac{((Re(\lambda))^2 + (Im(\lambda) - t)^2)^2}{4(Re(\lambda))^2((Re(\lambda))^2 + (Im(\lambda) - t)^2)^2} \\ &= \frac{1}{4(Re(\lambda))^2} \end{aligned}$$

Thus

$$\sigma(T) = \left\{\left|\omega - \frac{1}{2Re(\lambda)}\right| = \frac{1}{2Re(\lambda)} : \; Re(\lambda) > 0\right\}$$

For any $\mu \in \rho(T)$, we have $\mid \mu - \frac{1}{2\lambda} \mid > \frac{1}{2\lambda}$ which implies $Re(\lambda - \frac{1}{\mu}) > 0$

and thus $dist(\mu, \sigma(T)) = Re(\lambda - \frac{1}{\mu})$. Consequently,

$$\| R(\lambda - \frac{1}{\mu}, H) \| \leq \frac{M}{dist(\mu, \sigma(T))}$$

And from Lemma 5.1.7, we obtain

$$\| R(\mu, T) \| \leq \frac{1}{| \mu |} + \frac{1}{| \mu |^2} dist(\mu, \sigma(T))$$

It therefore follows from Theorem 5.1.5 that T has Bishop property (β). Moreover, the adjoint operator T^* satisfies $\sigma(T^*) = \sigma(T)$ and thus

$$\| R(\mu, T^*) \| \leq \frac{1}{\mu} + \frac{1}{\mu^2} dist(\mu, \sigma(T))$$

which indicates that T^* has Bishop's property (β). This implies that T has property (δ). Thus by Theorem 5.1.2 it follows that T is decomposable.

□

5.1.1 Hardy Spaces

Let $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}$ denote the unit disk of the complex plane and $H(\mathbb{D})$ denote the Fretchet space of functions analytic on $\mathbb{D}$. For $0 < p < \infty$, the hardy spaces on the unit disk, $H^p(\mathbb{D})$ are defined as $H^p(\mathbb{D}) = \{f \in H(\mathbb{D}) :\| f \|_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p d\theta < \infty\}$. We refer to [12] for the basic and comprehensive theory of Hardy spaces. In particular, it is important to note that every $f \in H^p(\mathbb{D})$, $0 < p < \infty$, has non tangential boundary values almost everywhere on $\partial\mathbb{D}$ and

$$\| f \|_{H^p(\mathbb{D})} = \left(\int_0^{2\pi} | f(e^{i\theta}) |^p \, d\theta \right)^{\frac{1}{p}}$$

Where we regard the boundary function as an extension of f. Moreover the growth condition for the functions in $H^p(\mathbb{D})$ is given by

$$| f(z) |^p \leq \frac{1}{1 - | z |^2} \| f \|^p_{H^p(D)}$$

$1 \leq p < \infty$, $f \in H^p(\mathbb{D})$.

We consider the following self analytic map $\varphi_t : \mathbb{D} \to \mathbb{D}$ given by

$$\varphi_t(z) = e^{-ct} z$$

for all $z \in \mathbb{D}$, $t > 0$. We define the corresponding weighted composition operators on $H^p(\mathbb{D})$ by

$$\begin{aligned} T_t f(z) &= (\varphi_t'(z))^\gamma f(\varphi_{t(z)}) \\ &= e^{-ct\gamma} f(e^{-ct} z) \end{aligned}$$

for all $f \in H^p(\mathbb{D})$, $\gamma = \frac{1}{p}$.

The following theorem gives both the semigroup and spectral properties of this group $\{T_t\}$ of composition operators.

Theorem 5.1.9

Let $H^p(\mathbb{D}), 1 \leq p \leq \infty$ be hardy space of the unit disk $\mathbb{D}$. Define a self analytic map $\varphi_t : \mathbb{D} \to \mathbb{D}$ by $\varphi_t(z) = e^{-ct} z$ and the corresponding weighted composition operator $T_t : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ by $T_t f(z) = e^{-ct\gamma} f(e^{-ct} z)$ where $c \in \mathbb{C}$, $t \geq 0$ and $\gamma = \frac{1}{p}$. Then the following hold:

(a) *$(T_t)_{t\in\mathbb{R}}$ is a group of isometries on $H^p(\mathbb{D})$*

(b) *$(T_t)_{t\in\mathbb{R}}$ is strongly continuous.*

(c) *The infinitesimal generator H of T_t is given by $Hf(z) = -cHf(z) - czf'(z)$ with the domain $dom(H) = \{f \in H^p(\mathbb{D}) : zf'(z) \in H^p(\mathbb{D})\}$*

(d) $\sigma(H) = \sigma_p(H) = \{-c(n + \frac{1}{p}) : n = 0, 1, 2, ...\}$

(e) *If $Re(c) = 0$, then $R(c, H)$ is compact, decomposable and a scalar type operator.*

PROOF. By definition and change of variables argument, we have

$$\begin{aligned} \| T_t f \|^p_{H^p(\mathbb{D})} &= \int_0^{2\pi} | (T_t f)e^{i\theta} |^p \, d\theta \\ &= \int_0^{2\pi} | f(\varphi_t(e^{i\theta})) | \varphi_t'(e^{i\theta}) i e^{i\theta} |^p \, d\theta. \end{aligned}$$

Let $\omega = \varphi_t(e^{i\theta})$, then $d\omega = \varphi_t'(e^{i\theta}) i e^{i\theta} d\theta$ and

$$\begin{aligned} \| T_t f \|^p_{H^p(\mathbb{D})} &= \int_0^{2\pi} | f(\omega) |^p \, d\omega \\ &= \| f \|^p_{H^p(\mathbb{D})} . \end{aligned}$$

This means that T_t is an isometry. Moreover, $T_t o T_s = T_{t+s}$ for all $s, t \in \mathbb{R}$ and $T_o = I$ where I is the identity operator. So $\{T_t\}_{t\in\mathbb{R}}$ is a group of isometries as desired.

To show that $\{(T_t)\}_{t\geq 0}$ is strongly continuous, it suffices to show that $\lim_{t\to 0} \| T_t f - f \|_p = 0$ for every $f \in H^p(\mathbb{D})$. Let $X(\mathbb{D})$ be the set containing all functions in $H^p(\mathbb{D})$ that are continuous on $\mathbb{D}$. Then $X(\mathbb{D})$ is dense in $H^p(\mathbb{D})$. Thus for $f \in H^p$ and arbitrary $\epsilon > 0$, there exists

$g \in X(\mathbb{D})$ such that $\| f - g \|_p < \epsilon$, then

$$\| T_t f - f \|_p \leq \| T_t f - T_t g \|_p + \| T_t g - g \|_p + \| g - f \|_p$$

$$= 2 \| f - g \|_p + \| T_t g - g \|_p$$

Now for all $g \in X(\mathbb{D})$, $T_t g(z) \to g(z)$ for all $g \in \partial D$ and by isometry of (T_t), we have $\| T_t g \|_p \to \| g \|_p$ Fatous lemma then gives $\| T_t g - g \|_p \to 0$. Thus $\| T_t f - f \|_p \leq 2\epsilon$, and hence (T_t) is strongly continuous.

By definition, the infinitesimal generator H of T_t is given by

$$\begin{aligned}
H(f) &= \lim_{t\to 0} \frac{T_t f - f}{t}, \; f \in D(H) \\
&= \lim_{t\to\infty} \frac{e^{-ct\gamma} f(e^{-ct}z) - f(z)}{t} \\
&= \frac{\partial}{\partial t} \left(e^{-ct\gamma} f(e^{-ct}z)\right) |_{t=0} \\
&= -c\gamma e^{-ct\gamma} f(e^{-ct}z) + e^{-ct\gamma} - (ce^{-ct} z f'(e^{-ct}z)) |_{t=0} \\
&= -c\gamma f(z) - czf'(z),
\end{aligned}$$

which implies that $D(H) \subseteq \{f \in H^p(\mathbb{D}) : zf'(z) \in H^p(\mathbb{D})$. Conversely, let $f \in H^p(\mathbb{D})$ such that $zf'(z) \in H^p(\mathbb{D})$. Then for $z \in \mathbb{D}$, we have

$$\begin{aligned}
T_t f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s} \left(e^{-cs\gamma} f(\varphi_s(z))\right) ds \\
&= \int_0^t \left(-c\gamma e^{-ctH} f(\varphi_s(z)) - czef'(e^{-ct}z)\right) \\
&= \int_0^t T_s(F) ds
\end{aligned}$$

where $F(z) = -c\gamma f(z) - czf'(z)$. Thus $\lim_{t\to 0} \frac{T_t f - f}{t} = \lim_{t\to 0} \frac{1}{t} \int_o^t T_s(F) ds$ Now for $F \in H^p(\mathbb{D})$ the limit exist and equal to F. Thus $D(H) \supseteq \{f \in$

$H^p(D) : zf'(z) \in H^p(D)\}$, as claimed.

To obtain the point spectrum of H, let λ be an eigenvalue and f be the corresponding eigenvector. Then the eigenvalue equation $Hf = \lambda f$ reduces to the differential equation

$$-cHf(z) - czf'(z) = \lambda f(z)$$

which is equivalent to

$$-czf'(z) = (\lambda + cH)f(z).$$

To solve the above ODE, we continue as follows;

$$\frac{f'(z)}{f(z)} = -\frac{1}{c}(\lambda + cH)\frac{1}{z}$$

$\Leftrightarrow$

$$\frac{df(z)}{f(z)} = -\frac{1}{c}(\lambda + cH)\frac{dz}{z}.$$

Therefore

$$\ln f(z) = -\frac{1}{c}(\lambda + cH)\ln z + C$$

and thus

$$f(z) = z^{-\frac{1}{c}(\lambda + cH)}$$

for $c \neq 0$. Since $z^{-\frac{1}{c}(\lambda + c\gamma)} \in H(\mathbb{D})$ if and only if $-\frac{1}{c}(\lambda + c\gamma) \in \mathbb{Z}_+$. That is $-(\gamma + \frac{\lambda}{c}) = n$, $n = 0, 1, 2, \ldots$. Hence $\sigma_p(H) = \{-c(n + \gamma) : n = 0, 1, 2, \ldots\}$ Clearly, if $Re(c) = 0$, then $c \in \rho(H)$ and therefore, the resolvent operator

$(c - H)^{-1}$ reduces to

$$R(c, H)f(z) = \frac{1}{cz}\int_0^z f(\xi)d\xi.$$

As remarked by Cowen and Macluer [5], such resolvents are compact and therefore

$$\sigma(H) = \sigma_p(H)$$

Now by Theorem 5.1.8, $R(c, H)$ is decomposable and hence of scalar type.

□

5.2 Application to Cauchy problems

In this section, we investigate some questions related to abstract Cauchy equations. Our interest is to apply the $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operators to analyze such equations.

Let H be a densely defined linear operator satisfying (1.4) such that (2.10) admits a unique exponentially bounded solutions $u(t)$ for every initial value $x \in D(H)$, then $u(t)$ belongs to the class of one parameter semi-group and the cosine families. An operator H on a Banach space X is the generator of k times integrated semigroup (where $k \in \mathbb{N}_o$) if there exist $w \geq 0$ and $S(.) : [0, \infty) \to B(E)$ a strongly continuous group such that (w, ∞) is contained in the resolvent set of H and

$$(\lambda I - H)^{-1}x = \lambda^k \int_0^\infty e^{-\lambda t}S(t)xdt \tag{5.2}$$

for all $x \in X$ and $\lambda > w$. The function $S(.)$ is called k-times integrated semigroup. It follows from the Hille Yosida theorem that one can characterize the operators H satisfying (1.4) for which (2.10) admits a unique solution given by a strongly continuous C_o semi group of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators acting on the Banach space X. The solution of (2.10) is given by $u(t, x) = T(t)x$ where $T(t) = e^{-tH}$ for $t \geq 0$ and $x \in X$. It follows that H is the infinitesimal generator of $u(t)$. On the other hand, some classes of Cauchy equations exist where the elements of $T(t)$ is not bounded operators, for example the Schrodinger operators acting on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $p \neq 2$. To deal with such problems, one needs to find larger sets of functions f giving rise to bounded operators in form of $e^{-tH}f(H)$ which is also a bounded solution of (2.10). This means that one must look for a suitable functional calculus involving H and e^{-tH}. We therefore resort to $\mathcal{U}$ functional calculus for $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators given by (1.5). More generally, $f(H)$ make sense if $f \in C_c^\infty(\mathbb{R}) \subseteq \mathcal{U}$ and H satisfy inequality in Theorem 2.4.12 for some $\alpha \geq 0$.

We now consider the abstract Cauchy equation given by (2.10). If a closed densely defined operator H has a resolvent in the half right plane and if $u(.)$ is an exponential bounded solution of (2.10) with $u(0) = x$, then the resolvent $R(\lambda, H)x$ is the Laplace transform of $u(.)$ i.e

$$R(\lambda, H)x = \int_0^\infty e^{-\lambda t} u(t)x dt \tag{5.3}$$

The following two theorems are consistent with the $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators discussed in this thesis and their proofs can be found in [26].

Theorem 5.2.1

Let H be a linear operator on a Banach space X. If there exist constants w and C such that $R(\lambda, H)$ exist and satisfy

$$| R(\lambda, H) |\leq C(1+ | u |)^k \tag{5.4}$$

for some $-1 \leq k$ and for all $u \in \mathbb{C}$ with $\mathrm{R}(\lambda) > w$ then (2.10) has a unique solution $u(.)$ for every $x \in D(H)$ such that $| u(t) |\leq Ce^{pt} \| x \|$ for $p > w$

Theorem 5.2.2

Let H be a linear operator with nonempty resolvent set. If (2.10) has a solution $u(.)$, with $u(0) = x$ such that $| u(t) |\leq Ce^{pt}$ for some constants C, p then for every $\lambda \in \rho(H)$ with $R(\lambda) > p$ we've

$$R(\lambda, H)x = \int_0^\infty e^{\lambda t} u(t) x dt \tag{5.5}$$

The following is an immediate consequence of Theorem 5.2.1

Corollary 5.2.3

If H is of $(0,1)$ type $\mathbb{R}$ with $C = 1$, then (5.4) reduces to

$$| R(\lambda, H) |\leq 1 \tag{5.6}$$

for $k = 0$ and (2.10) has a unique solution $u(t)$ which is bounded above by 1 for $\| x \|= 1$. In that case

$$R(\lambda, H)x = \int_0^\infty e^{-\lambda t} u(t) x dt \leq \int_0^\infty e^{-\lambda t} dt \leq \frac{1}{\lambda} \tag{5.7}$$

in particular, if $u(t)$ is a contraction then the solution $u(t)$ satisfying (2.10) is bounded above by 1.

Definition 5.2.4

The Schwartz space $S(\mathbb{R}^n)$ of rapidly decreasing smooth functions consists of all $f \in C^\infty(\mathbb{R}^n)$ satisfying

$$lim_{|x|\to\infty} P(x) \frac{\partial^{\alpha_1+...,+\alpha_n}}{\partial x_1^{\alpha_1} ..., \partial x_n^{\alpha_n}}(x) = 0$$

for each polynomial P and each partial derivative as indicated above and $(\alpha_1, ..., \alpha_n \in \{0, 1, 2, ...\})$

REMARK 5.2.5

We note that $C_c^\infty(\mathbb{R}) \subset S(\mathbb{R}^n)$. Here, $f \in C_c^\infty(\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R})$ and f has compact support. Also $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and in $C_o(\mathbb{R}^n)$, the continuous functions on $\mathbb{R}^n$ that vanish at infinity; hence $S(\mathbb{R}^n)$ is also dense in these spaces.

We now define the following family $S(\beta)$, $\beta \in \mathbb{R}$ found in [18] as follows:

Definition 5.2.6

$f \in S(\beta)$ if $f \in C_c^\infty(\mathbb{R})$ and $f(\lambda)$ has an asymptotic expansion in λ^{-1} as $\lambda \to \infty$ in the following sense. For any $N > 0$

$$f(\lambda) = \sum_{K=0}^{N} a_k \lambda^{-\beta-k} + \gamma_N(\lambda) \tag{5.8}$$

$\lambda \geq 1$ and where $\gamma_N(\lambda)$ satisfy

$$| \left(\frac{d}{d\lambda} \gamma_N(\lambda) \right) | \leq C_{N_k}(1+ | \lambda |)^{-\beta-N-1} \tag{5.9}$$

for all $\lambda \geq 1\, k = 0, 1, 2...,$

If $\beta = 0$ then (5.8) reduces to

$$f(\lambda) = \sum_{K=0}^{N} a_k \lambda^{-k} + \gamma_N(\lambda) \tag{5.10}$$

and $\gamma_N(\lambda)$ satisfy

$$\mid \left(\frac{d}{d\lambda}\gamma_N(\lambda)\right) \mid \leq C_{N_k}(1+ \mid \lambda \mid)^{-N-1} \tag{5.11}$$

for all $\lambda \geq 1\ k = 0, 1, 2...,$

We now state the following theorems whose proofs can be found in [18].

Theorem 5.2.7

Let $1 \leq p \leq \infty$ and let $f \in S(\infty)$. Then $e^{-itH}f(H)$ is bounded in $L^p(\mathbb{R}^N)$ for $t \in \mathbb{R}$. Moreover, for $\beta > N \mid 1/p - 1/2 \mid$,

$$\| e^{-itH}f(H) \| \leq C(1+ \mid t \mid)^{\beta},\ t \in \mathbb{R} \tag{5.12}$$

Theorem 5.2.8

Suppose $N \leq 3$ and let $1 \leq p \leq \infty$. If $f \in S(\beta)$ for some $\beta > 2 + N/4$ then

$$\| e^{-itH}f(H) \| \leq C(1+ \mid t \mid)^{N|1/p-1/2|},\ t \in \mathbb{R} \tag{5.13}$$

Theorem 5.2.9

Let H be a schrodinger operator on $L^p(\mathbb{R}^N)$ then H is of $(\alpha, \alpha+1)'$ type $\mathbb{R}$ for $\alpha := N \mid 1/p - 1/2 \mid$.

REMARK 5.2.10

Theorem 5.2.9 holds whenever we replace $\langle\rangle$ by $|\,|$ in (1.4) and it is stronger than (1.4) since $| z | \leq < z >$ for all $z \in \mathbb{C}$. Therefore $(\alpha, \alpha + 1)'$ type $\mathbb{R}$ implies $(\alpha, \alpha + 1)$ type $\mathbb{R}$.

REMARK 5.2.11

The definitions above and theorems are consistent with those of $(\alpha, \alpha+1)$ type $\mathbb{R}$ discussed in this thesis.

We now consider the abstract Cauchy equation given by;

$$\begin{cases} u'(t) = -iHu(t), & t \geq 0; \\ u(0) = x, & x \in C_c^\infty(\mathbb{R}). \end{cases} \tag{5.14}$$

where H satisfies (1.4). Our first result is given by the following theorem.

Theorem 5.2.12

Let H be $(\alpha, \alpha + 1)$ type $\mathbb{R}$ operator, then $u(t) \in C_c^\infty(\mathbb{R})$ is a solution of (5.14) provided that $u(t)$ satisfies theorem 2.4.12

PROOF. Let $u(t) \in C_c^\infty(\mathbb{R})$ such that $u(t) = e^{-iHt}$ for $t \in \mathbb{R}$ and H is of $(\alpha, \alpha + 1)$ type $\mathbb{R}$, then $u(t)$ satisfies Theorem 2.4.12. It follows that $u'(t) = -iHu(t)$ satisfies (5.14) and $u(0)x = x$ for each $x \in \mathbb{R}$ and therefore, $u(t)$ is a solution of (5.14). Now since H has a resolvent lying on the right half plane, and u(t) is a solution of (5.14) with $u(0) = x$ and $u(t)$ satisfying Theorem 2.4.12, we have that

$$\begin{aligned} R(\lambda, -iH)x &= \int_0^\infty e^{\lambda t} u(t) x dt \\ &= \int_0^\infty e^{\lambda t} e^{-iHt} x dt \\ &\leq C(1+ | t |)^\alpha \end{aligned}$$

for all $t \in \mathbb{R}$ and some $\alpha \geq 0$. It follows that u(t) is the unique solution of (5.14) and that $R(\lambda, -iH)$ is the inverse Laplace transform of $u(t)$. □

Our second result is given via the following theorem.

Theorem 5.2.13

Let H be $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator, and $u(t) \in C_c^\infty(\mathbb{R})$ be a convergent solution of (5.14) then $u(t)f(H)$ is also a convergent solution of (5.14) for every $f \in C_c^\infty(\mathbb{R})$

Proof. Suppose that Theorem 5.2.7 holds and $f \in C_c^\infty(\mathbb{R})$, then $f(H)$ can be extended to the space $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$. Letting $\alpha = 0$, it is also shown in [18], that if $1 \leq p \leq \infty$ and $\alpha \to \infty$ then $u(t)f(H)$ is bounded in $L^p(\mathbb{R})$ for $t \in \mathbb{R}$ and satisfy inequality in theorem 2.4.12. In particular, if $\alpha = N \mid 1/p - 1/2 \mid$, then

$$\| u(t)f(H) \| \leq C(1+ \mid t \mid)^{N|1/p-1/2|}, t \in \mathbb{R} \tag{5.15}$$

Since (5.15) and inequality of Theorem 2.4.12 have the same bound, and $u(t)$ is a convergent solution of (5.14), it follows that $u(t)f(H)$ is also a convergent a solution of (5.14) and this completes our proof. □

Chapter 6

Summary and Recommendations

6.1 Summary

In this thesis, we applied the properties of scalar type operators to give our characterization. We used two major aspects in our characterization, that is the $\mathcal{U}$ functional calculus and semi group theory of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators. Our contributions in the Characterization of scalar type operators using functional calculus is given by Theorem 4.2.2 in which we applied the properties of the semi-group theory and the $\mathcal{U}$ functional calculus to show that a linear operator H acting on a Hilbert space $\mathcal{H}$ is scalar type if it is of $(0, 1)$ type $\mathbb{R}$ and it admits a bounded functional calculus. In Theorem 4.2.4, we have applied the properties of the resolvent in Theorem 2.4.7 and the properties of self adjoint operators to show that if H is a scalar type operator on a Hilbert space $\mathcal{H}$, then its adjoint H^* is also a scalar type operator on $\mathcal{H}$ and that H^* admits a $\mathcal{U}$ functional calculus. In application of scalar type operators to decomposability, we have shown using Theorem 5.1.9, that if $T(t)$ is a group of symmetries on a Hardy space $H^p(\mathbb{D})$ and $c \in \mathbb{C}$, such that $Re(c) = 0$ and $R(c, H)$ is decomposable, then H is scalar type. In Theorem 5.2.12 we have shown that if an operator H is of $\alpha, \alpha + 1$ type $\mathbb{R}$, then $u(t)$ is a solution of (5.14) if and only if $u(t)$ satisfy Theorem 2.4.2. In Theorem 5.2.13, we have shown that if Theorem 5.2.12 holds, then u(t)f(H) is also a solution of (5.14).

6.2 Recommendations

From the results of this study, we recommend the following for further research;

1. Extension of our characterization to two commuting scalar type operators.

2. Application of scalar type operators to decomposability can be extended to other spaces of analytic functions like Dirichlet spaces, Bergman spaces, Bloch spaces among others.

3. It would also be interesting to consider the study of groups of isometries and the resulting integral operators on other Hardy spaces apart from Hardy spaces of the unit disk $\mathbb{D}$ that has been considered in this study. For example, Hardy spaces of the upper half of the complex plane. subsequently an investigation of the decomposability of the resulting resolvents will be necessary.

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