APPLICATION OF FACTORISATION AND BOOST TRANSFORMATIONS PROCEDURES ON SINGLE FIELD INFLATIONARY COSMOLOGICAL PERTURBATIONS

BY

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN PHYSICS

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DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

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ACKNOWLEDGEMENTS

I give honor and glory to Almighty God for giving me good health and intellectual power to successfully complete this study. May He continue protecting and guiding me all the days of my life. Amen.

I would like to thank my family for their economic and emotional support especially my parents who always encouraged me to pursue a Masters Degree and making me believe in myself and my potential.

I would like to extend my gratitude and sincere thanks to my supervisors **Prof. Joseph Akeyo Omolo** and **Dr. Boniface Otieno Ndinya**, for their constant motivation and support during the course of my work. I truly appreciate and value their esteemed guidance and encouragement from the beginning to the end of this thesis. I would also wish to thank them for their valuable suggestions and comments during this project period.

I am very thankful to my teachers **Prof. Herrick Othieno, Prof. Andrew Oduor, Prof. Reccab Ochieng** and **Mr. Jack Okumu** for providing solid background for my study and research thereafter.

I would like to thank the staff of the Department of Physics and Materials Science, Maseno University, for their constant support and providing place to work during project period. I would also like to extend my gratitude to my friends who have been with me during the project period and have helped me with valuable ideas.

DEDICATION

To the memory of our father,
Our Guardian Angel
Mr. Lazarus Dom Odingo Okombo (Major)
1956-2014
May his Soul Rest in Eternal Peace

ABSTRACT

Inflation is a period of an accelerated expansion of the universe. Cosmological perturbations are created by the amplification of quantum vacuum fluctuations of matter and metric perturbations during inflation. The equation of dynamics governing the evolution of cosmological perturbations within the cosmological model in a single field inflationary scenario takes the form of a standard non-linear second-order differential equation, whose exact analytical solution has not been obtained to date. The various methods of approximation that have been used in solving this equation of dynamics have varied limitations that include: inadequate error control; difficulty in improving the accuracy beyond the leading order (are not systematically extendable); complicated/tedious mathematical formulations; and series expansions that may be also divergent at some order. This study provides a systematically extendable method of approximation for the study of single field cosmological perturbations during inflation, which removes the divergence in the Wentzel-Kramers-Brillouin (WKB) approximation, based on a factorization and boost transformation procedures up to zeroth-order. The equation of dynamic is factorized and then converted into a matrix equation with its corresponding Hamiltonian. By using appropriately defined boost transformation operator, the resultant matrix equation undergoes successive boost transformations along suitable axes to new dynamical frames of high accuracy levels, characterized by an approximation parameter that becomes smaller with increasing number of boost transformations and is safely neglected at the highest level of approximation (accuracy). Diagonalization of the boost frame Hamiltonian leads to a simple analytical solution of the boost frame matrix equation through direct integration, and once the time evolution operator in the form obtained through the diagonalization procedure, is used into the analytical solution obtained, the desired general solution of the equation of dynamics governing the evolution of cosmological perturbations in single field inflationary scenario follow easily up to n^{th} −order.The zeroth-order derivative of the approximation parameter produces an expression that is more exact and similar to the standard WKB approximation parameter Q, though with different co-efficient. Each order of approximation provides an amplitude-modulated "plane" mode function specified by a renormalized time-dependent frequency, that facilitates exact evaluation of the phase accumulation integral for various forms of the potential U. The zeroth-order solution is exactly the leading order/first- order standard WKB solution, and takes exactly the same form of the assumed solution (ansatz) in the standard WKB approximation. Furthermore, it does not require any matching condition about any particular point, that is to say, a turning point as outlined in the results for the WKB mode function and therefore the issue of divergence at a turning point as encountered in the WKB approximation does not arise in the approximation procedure developed in this study. This goes a long way to improving the understanding of inflationary perturbations. The general expressions for the zeroth-order power spectrum constitute the main results of this study.

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LIST OF ABBREVIATIONS

WKB Approximation – Wentzel-Kramers-Billouin approximation

CMB – Cosmic Microwave Background

FRW - Friedman-Lemaitre-Robertson-Walker

WMAP - Wilkinson Microwave Anistropy Probe

BOOMERanG - Balloon Observations of Millimetric Extragalatictic Radiation ANd Geophysics

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CHAPTER ONE

INTRODUCTION

1.1 Background Information

The inflationary cosmological perturbation is most likely the most important matter perturbation, since there is evidence that it causes both the Cosmic Microwave Background (CMB) temperature fluctuations and structure formation in the universe. Recent observations of the relic photons created when the universe was hot and dense, which do survive today with a much lower temperature since their wavelength is increased due to the expansion of the universe (inflation), [1] and of galaxy distribution in real and red shift space (light emitted from a far away galaxy needs more time to reach us than light from a galaxy in our local group; and as a result, the far away galaxy is observed in a much earlier stage of its life than the nearby one), have dramatically improved our knowledge of the large-scale structure of the universe. Results from these observations have been largely consistent with the inflationary paradigm of cosmology and none of them would have been possible unless the celestial objects were emitting light, [2]. Cosmology describes the structure and evolution of the universe on the largest scales. Modern models of the Universe, including the Hot Big Bang [5] are based on an important assumption, namely, the Cosmological Principle, which states that; the Universe is, on the largest scales, nearly perfectly isotropic and homogenous. In the framework of General Relativity, a universe that is both isotropic and homogenous is described by the Friedman-Lemaitre-Robertson-Walker metric

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\Theta^2 + sin^2\Theta d\varphi^2)\right] \tag{1}$$

where $a(t)$ is the scale factor of the Friedman-Lemaitre-Robertson-Walker metric [3], and $k = -1,\ 0, 1,$ is the curvature signature. All three models are without boundary; the positively curved model is finite and "curves" back on itself; the negatively curved and flat models are infinite in extent. The coordinates r, θ and φ, are referred to as commoving coordinates [3]

Inflation predicts a spectrum of metric perturbations in the scalar (density) and tensor (gravitational wave) sectors, the vector component being naturally suppressed [4] and are determined on this background by the two functions respectively denoted as $\square = \square_S = -2z_S\zeta$, where ζ is the Bardeen's hyper surface-independent quantity [19] or $\square_S = aQ$ (Q is the Mukhanov variable) [9], and $\square = \square_T = z_T h = ah$ where h is the amplitude of gravitational waves, which satisfy a one-dimensional Schrödinger-like equation [6]

$$\left[\frac{d^2}{d\eta^2} + \omega^2(k,\eta)\right] \square(\eta) = 0. \tag{2}$$

The effective time dependent frequency is given by the general expression,

$$\omega^2(k,\eta) = k^2 - U(\eta) = k^2 - \frac{z''}{z} \quad , \tag{3}$$

where k is the wave number (primes denote derivative with respect to conformal time η) and

$$z_S = a\sqrt{2 - \frac{aa''}{a'^2}}\,, \tag{4a}$$

for scalar (density perturbations), and

$$z_T = a\,, \tag{4b}$$

for tensor (gravitational wave perturbations). Equation (2) is the equation governing the evolution of cosmological perturbations in a single field inflationary scenario, and its physical interpretation is parametric amplification of perturbations during inflation. The perturbations are generated due to the amplification of quantum vacuum fluctuations by the dynamics of the background space-time, and the scale factor plays the role of a 'pump' field and the 'interaction' between the background and the perturbations is described by the potential $U(\eta)$ [4].

The mode function ⍰(η) in equation (2) is a quantity of interest because the power spectra of density perturbations and gravitational waves, which are observables, directly involve them. Equation (2) for the mode functions ⍰(η) must be solved together with the condition that the modes are initially plane waves for wavelengths much shorter than the Hubble radius [7].

$$\lim_{\frac{k}{aH}\to\infty} ⍰_{s,T}(\eta) = \pm\frac{\sqrt{8\pi}}{m_{pl}}\frac{e^{-i(\eta-\eta_i)}}{\sqrt{2k}}\,, \tag{5}$$

where η_i is an arbitrary time at the beginning of the inflation and m_{pl} is the Planck's mass. This initial condition corresponds to the fact that, initially the modes are sub horizon and therefore do not feel the curvature of space-time. As a consequence, they are described by plane waves. If the initial quantum state is the vacuum state, then the statistical properties of the perturbations are entirely characterized by the two-point correlation function, that is to say, the power spectrum. The dimensionless power spectra for scalar and tensor fluctuations are governed by [6]

$$P_\zeta(k) = \frac{k^3}{8\pi^2}\left|\frac{⍰_S}{Z_S}\right|^2 \qquad ; \qquad P_h(k) = \frac{2k^3}{\pi^2}\left|\frac{⍰_T}{Z_T}\right|^2. \tag{6}$$

The spectral indices and their running's are defined by the co-efficient of Taylor expansions of the power spectra with respect to $\ln k$, evaluated at an arbitrary pivot scale k_* given by

$$n_S - 1 = \frac{dln\, P_\zeta}{dlnk}|_{k=k_*} \qquad ; \qquad n_T = \frac{dln\, P_h}{dlnk}|_{k=k_*}. \tag{7}$$

The two following expressions define the "running" of these indices,

$$\propto_S = \frac{d^2 \ln P_\zeta}{d(\ln k)^2}|_{k=k_*} \qquad ; \qquad \propto_T = \frac{d^2 \ln P_h}{d(\ln k)^2}|_{k=k_*}. \tag{8}$$

In order to calculate the perturbation spectra for these quantum vacuum fluctuations, three steps are necessary: the dynamics of the background space-time must be determined; the mode equations for scalar and tensor perturbations must be solved; and finally the power spectra themselves must be calculated as functions of a wave number k [1]. The power spectra is a variable that quantifies the variance of the perturbations as a function of the commoving wavenumber k. As for its physical importance, the scalar and tensor perturbations of inflation are the contact between theory and observation. According to [20], by confronting theoretical predictions with observations, cosmologists aim to reduce the number of viable inflation theories or even pin down the correct model which will ultimately test Ultra-high energy Physics beyond reach of any earth-bound accelerator. The equations governing the evolution of scalar and tensor perturbations in single field inflationary models can be solved numerically, so that comparison with available or future data will tell us which models satisfactorily represent the time evolution of the Universe. Since exact solutions are not available for cosmological perturbations in general inflationary models (except for the case of an exponential potential), approximation methods are welcome [15].

1.2 Statement of the problem

Traditionally the method of choice for inflationary cosmology is the slow-roll approximation subject to an infinite number of convergence conditions, or on numerical integration. Recently, semi-classical methods such as: the WKB approximation method; the phase integral method; and the method of uniform approximation have been applied. The various methods of approximation that have been used in solving the equations governing the evolution of scalar and tensor perturbations in single field inflationary cosmological perturbations have varied limitations that include: inadequate error control; difficulty in improving the accuracy beyond the leading order (are not systematically extendable); complicated/tedious mathematical formulations; and series expansions that may be also divergent at some order. Therefore, a method of approximation for inflationary cosmology that is able to improve on any of the limitations is necessary.

1.3 Objectives of the study

The objectives of the study are:

1. Develop a systematic method of approximation for the study of cosmological perturbations during inflation, characterized by an approximation parameter which becomes progressively smaller with increasing number of successive boost transformation.
2. Compute approximate expressions for the power spectra of scalar (density) perturbations and tensor (gravitational wave) perturbations in a single field inflationary scenario, and compare the results obtained with the Wentzel-Kramers-Brillouin (WKB) approximation results.

1.4 Justification of the study

The study provides a method of approximation for the study of cosmological perturbations during inflation. The procedure does not make restrictive assumptions, is systematically extendable, and is valid for any form of the potential $U(\eta)$. This allows for the determination of the spectra beyond the leading order.

1.5 Significance of the Study

The study has provided a way of removing the divergence at the turning point in the WKB approximation procedure, by applying an approximation method that is normally used in quantum mechanics to the study of cosmological perturbations during inflation. This will go a long way in improving not only the understanding of inflationary cosmology but also of quantum mechanics.

1.6 Limitations of the study

The limitations of this study include;

1. The study only focuses on single inflation field models and not to other models such as the multiple field models, nontrivial potentials model etc.
2. Apart from the standard slow-roll approximation, other approximation schemes carried out in previous studies allow determination of the spectra beyond leading order and are subject to different limitations. Comparison of results obtained from the approach developed in this thesis is only limited to the results obtained from the study of cosmological perturbations during inflation through the Wentzel-Kramers-Brillouin (WKB) approximation.

CHAPTER TWO

LITERATURE REVIEW

2.1. Cosmic Evolution

One of the most crucial questions asked by scientists and philosophers since ancient Greece is whether the Universe is finite, and if so whether it is of constant size. Infact, Freidman found that according to General Relativity the Universe should either be expanding or contracting [10]. The first observational evidence of the time evolution of the universe was the discovery made by E. Hubble in 1929, that all galaxies observed today move away from our own galaxy with a velocity v proportional to their distance d_l, $(v = Hd_l)$. This is commonly known as the Hubble's law. Given the cosmological principal, this discovery leads to the idea that the Universe is expanding with time. One can think of a Universe filled with galaxies and clusters of galaxies that do not expand themselves because of the gravitational attraction, but which recede from one another because of the expansion of the Universe. Extrapolating back in time, one discovers a Universe that was smaller and more dense. Extrapolating even further back in time, one reaches a singularity where all the matter in the Universe was concentrated at a point at time $t = 0$. This is called the Big Bang. The standard Big Bang theory assumes that the Universe today emerged from such a singularity, and initially rapidly expanded as an extremely hot and dense system which was made up from matter and radiation [5]. From there on, the evolution of the Universe is governed by laws of Physics at high energies. The expansion rate is gradually decreasing as time goes on, the energy, temperature, and the density of the Universe decrease.

In the early stages of the Big Bang, most of the energy was in the form of radiation, and that radiation was the dominant influence in the expansion of the universe. Later with cooling from the expansion, the roles of matter and radiation changed and the universe entered a matter-dominated era. Recently, results suggest that we have directly entered an era dominated by dark-energy [8].

The evolution of the universe depends on the single function (the scale factor $a(t)$ of the Universe), whose form is dictated by the matter content of the universe through the Einstein's field equation [2]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \; . \tag{9a}$$

Here, $G_{\mu\nu}$ is the Einstein's tensor, $g_{\mu\nu}$ is the metric of the manifold in which the equations apply, $\square_{\alpha}$ is the macroscopic speed of the medium, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci Scalar

$$R_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \quad ; \quad R \equiv g^{\mu\nu} R_{\mu\nu} \; , \tag{9b}$$

where

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right], \tag{9c}$$

are the Christofell symbols, $T_{\mu\nu}$ is the energy-momentum tensor and G is the Universal gravitation constant. The energy-momentum tensor incorporates matter into the Einstein equations, and the form given here is true for a perfect fluid [9].

The evolution of the scale factor $a(t)$ follows from the Einstein's field equations. The (0,0) component of the Einstein's equation gives [2]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \, , \tag{10}$$

where

$$H = \frac{\dot{a}}{a} = \frac{\mathcal{H}}{a} \qquad ; \qquad \mathcal{H} = \frac{a^{'}}{a}\,, \tag{11}$$

is the Hubble parameter and $\mathcal{H}$, is the Hubble rate, a dot meaning derivative with respect to cosmic time $\frac{dt}{d\eta} = a(\eta)$ while a prime stands for a derivative with respect to conformal time.

Differentiating equation (10) with respect to time, and using the mass conservation equation,

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{12}$$

we find

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{13}$$

where ρ and p are the proper energy density and pressure in the fluid rest frame.

The two quantities p and ρ are linked by the equation of state, $p = \omega(\rho)$, with a constant ω. For the different values of $\omega = 0, \frac{1}{3},$ and -1, we get the equations of state of matter, radiation, and vacuum energy respectively [2].

Equations (10) and (13) are the Friedman equations and are fundamental in studies of cosmology, and are used to derive the behaviour of the scale factor for different eras of the Universe, through the cosmological pressure and density of each era. Equation (10) relates the curvature k of the universe to the expansion rate $\frac{\dot{a}}{a}$ and the density ρ, while equation (13), relates the acceleration $\frac{\ddot{a}}{a}$ to the density plus three times the pressure p. The Friedman equations together with the equation of state form a complete system of equations that determine the unknown functions $a(t)$ and $\rho(t)$.

2.2. Inflation

Differentiating the Freidman equation for a flat universe obtained on using $k = 0$ in equation (10) with respect to time, and replacing $\dot{\rho}$ according to the energy conservation equation in equation (12), we obtain

$$\ddot{a} = -\sqrt{\frac{8\pi G}{3}}\frac{\dot{a}}{2\rho^{\frac{1}{2}}}(\rho + 3p). \tag{14}$$

As long as the universe is expanding $\dot{a}$ is positive. So, the equation for accelerated expansion is simply

$$\rho + 3p < 0. \tag{15}$$

From equation (13) and equation (14), one sees that any form of matter such that $\rho + 3p < 0$ will cause an acceleration of the scale factor (exponential growth/decay). This is only true, of course, if the matter component satisfying $\rho + 3p < 0$ is the dominant one [9]. The energy density is always positive but, in some situations, the pressure can be negative (negative pressure produces a repulsive form of gravity, because it leads to an acceleration of the scale factor a, if the pressure term $3p$ dominates over the energy density ρ) and the inequality $\rho + 3p < 0$ may be realized. This simple remark is at the heart of the inflationary scenario. If the strong energy dominance condition, $\rho + 3p > 0$, is satisfied, then from equation (13) and equation (14), we see that $\ddot{a} < 0$ and gravity decelerates the expansion (form points to oscillatory behaviour with time varying frequency). Therefore, for an accelerated expansion, the strong energy dominance condition must be violated.

The possibility of an early exponential expansion was first noticed by Starobinsky in 1979-1980 [11], but at first it attracted little attention. It was Guth in 1981 [12], who noted that an inflationary period could solve the flatness and the horizon problems. In the model proposed by Guth, known as old inflation, a scalar field is trapped at the origin in a local minimum of its potential and hence the universe is dominated by the field's vacuum energy. Inflation ends when the field tunnels through the barrier and descends quickly to the minimum of the potential. However, this model could not provide sufficient reheating and it was soon abandoned.

In 1982, A. D Linde and Albrecht Steinhardt [18], proposed the new inflation model. Assuming a phase transition, the inflation is initially situated on a maximum of its potential at the origin. The field starts slowly rolling down the rather flat potential. Inflation ends when it reaches its minimum and starts oscillating around it, reheating the universe. Although this type of potential was abandoned due to observational constraints, new inflation first introduced the concept of slow-roll inflation.

Later on, chaotic inflation was proposed, during which the field rolls down the origin in a $\phi^2 \; or \; \phi^4$ potential [13]. Its name derives from the chaotic initial conditions that are used to explain the needed large initial value of the field. During chaotic inflation the fields magnitude is of the order of the Planck's mass m_{pl}, and hence it is not easy to make connection with particle theories. However, because of their simplicity, monomial models became the favoured paradigms of inflation. Many models have been built since then; among these we distinguish single scalar field inflationary models.

2.3. The cosmological scale factor

The universe is understood to have evolved through various epochs, with each epoch named after the component that has a dominant contribution to the total energy density. The relative expansion of the universe is parametrized by the dimensionless scale factor $a(t)$, [12].

According to [8], in the early stages of the Big Bang, most of the energy was in the form of radiation, and that radiation was the dominant influence in the expansion of the universe. Later, with cooling from the expansion the roles of matter and radiation changed and the universe entered a matter-dominated era. Recently, results suggest that we have directly entered an era dominated by dark energy. The scale factor is obtained by solving the Freidman equations. For a spatially flat universe, $k = 0$ and the solution for the scale factor is [8]

$$a(t) = a_o t^{\frac{2}{3(\omega+1)}} \ , \tag{16a}$$

where a_o is some integration constant to be fixed by the choice of the initial conditions and t is the physical time, $dt = a(t)dr$. This family of solutions labelled ω in equation (16a) is extremely important for cosmology [8], for $= 0, \frac{1}{3}$, and -1 , we get the scale for matter, radiation and vacuum energy dominated universe respectively.

In commoving coordinates η is the conformal time defined as

$$d\eta = \frac{dt}{a(t)} \qquad \Longrightarrow \qquad \eta = \int \frac{dt}{a(t)} \ . \tag{16b}$$

The conformal time can be expressed as a function of the scale factor; for the universe with radiation component

$$a(t) \sim t^{\frac{1}{2}} \quad \Longrightarrow \quad a(\eta) \sim \eta\,, \tag{16c}$$

while for non-relativistic matter

$$a(t) \sim t^{\frac{2}{3}} \quad \Longrightarrow \quad a(\eta) \sim \eta^2\,. \tag{16d}$$

2.4. Single scalar field inflation

During inflation the pressure should be negative and smaller than $-\frac{\rho}{3}$. A. Liddle and D. H Guth [12] noted that a cosmological constant Λ could do the job since it has $p = -\rho$. During a fully Λ-dominated stage, the Hubble parameter remains constant: one has a De Sitter (exponential) inflation. The problem is that a cosmological constant never decays, so the inflation will be indefinite. If we want inflation to end, "something must happen", so there must be an arrow of time. Therefore, the type of matter responsible for inflation cannot be exactly in equilibrium.

The simplest model for a successful inflation compatible with the symmetries of the Friedman-Lemaitre-Robertson-Walker metric is to consider models where inflation was caused by the dynamics of a scalar field (called the inflaton) in a relatively featureless potential, evolving in an effectively friction-dominated "slow-roll" regime, and described by the corresponding Langrangian

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_o \partial_\nu \varphi_o + V(\varphi_o)\right], \tag{17}$$

where $V(\varphi_o)$ is the potential [2]. Because it is rolling, there is an arrow of time, and something can happen that will end inflation. But because the valley is very flat and the rolling is very slow, the field can be seen at any time as in an "instantaneous vacuum state" sharing almost the properties of a true vacuum state: in particular, the energy of the field is diluted very slowly, and the pressure is very close to $-\rho$.

In order to have a successful inflationary phase, the Langrangian in equation (17) must satisfy some constraints. The stress-energy tensor can be written as

$$T_{\mu v} = \partial_\mu \varphi_o \partial_v \varphi_o - g_{\mu v}\left[\frac{1}{2} g^{\alpha\beta}\, \partial_\alpha \varphi_o \partial_\beta \varphi_o + V(\varphi_o)\right]. \tag{18}$$

From this expression, one sees that the scalar field can also be viewed as a perfect fluid. The energy density and the pressure are defined according to, $T_0^0 = -\rho$, and $T_j^i = P\delta_j^i$, and reads [16],

$$\rho = \frac{1}{2}\frac{(\varphi_o')^2}{a^2} + V(\varphi_o), \tag{19a}$$

$$P = \frac{1}{2}\frac{(\varphi_o')^2}{a^2} - V(\varphi_o), \tag{19b}$$

where the first and second terms in equations (19a) and (19b) are the kinetic energy density and the potential energy density respectively.

Substituting equation (19a) and (19b) into the condition for inflation given in equation (15), we find

$$\frac{1}{2}\frac{(\varphi_o')^2}{a^2} + V(\varphi_o) + \frac{3}{2}\frac{(\varphi_o')^2}{a^2} - 3V(\varphi_o) < 0 \quad \Longrightarrow \frac{(\varphi_o')^2}{a^2} < V(\varphi_o). \tag{20}$$

Hence, during inflation, the potential energy of the inflaton dominates the kinetic energy of the inflaton. For the above condition to hold the potential is required to be flat enough and hence the scalar field is required to slowly roll. If the potential has a minimum, the above condition can also predict an ending to the period of inflation [9].

2.5. The WKB Approximations

A method for predicting inflationary cosmological perturbations to first order in the adiabatic (or "semi-classical) expansion based on the standard WKB approximation, (named after G. Wentzel, H.A Kramers, and L. Brillouin) was presented by [6], by first assuming a solution in form of WKB ansatz $\square_{WKB}$ given by

$$\square_{WKB}(k,\eta) = \frac{1}{\sqrt{\omega(k,\eta)}} e^{\pm i \int^{\eta} \omega(k,\tau) d\tau}, \tag{21}$$

representing the leading order term of the semi-classical expansion. Substituting $\square = \square_{WKB}$ from equation (21) into equation (2) provides a differential equation

$$\square''_{WKB}(k,\eta) + \omega^2(k,\eta)\left[1 - \frac{Q(k,\eta)}{\omega^2(k,\eta)}\right]\square_{WKB}(k,\eta) = 0, \tag{22}$$

which differs from equation (2) by the term $Q\square_{WKB}$ where the quantity $Q(k,\eta)$ is obtained as

$$Q(k,\eta) = \frac{3}{4}\frac{(\omega')^2}{\omega^2} - \frac{\omega''}{2\omega}, \tag{23}$$

depending on the time dependent frequency $\omega(k,\eta)$.

Comparing equations (2) and (22), it is concluded that the WKB mode function $\square_{WKB}$ is a good approximation of the actual mode function $\square$, if the following condition is satisfied,

$$\Delta \coloneqq \left|\frac{Q}{\omega^2}\right| \ll 1 \ \Longrightarrow 1 - \frac{Q}{\omega^2} \simeq 1, \tag{24}$$

and therefore

$$\square''_{WKB}(k,\eta) + \omega^2 \square_{WKB} \simeq 0. \tag{25}$$

On introducing a Langer transformation [6] defined by

$$x \coloneqq ln\left(\frac{k}{aH}\right) \qquad , \quad \chi \coloneqq (1-\epsilon_1)^{\frac{1}{2}}\, e^{-\frac{x}{2}}\, \square, \tag{26}$$

in order to improve the accuracy of the calculations, equation (22) is re-written in the form

$$\ddot{\chi} + \omega(x)^2\chi = 0, \tag{27}$$

where the (new) frequency $\omega(x)$ in general vanishes at the classical "turning point" $x = x_*$ [20].

The first-order WKB approximation of the equation $\ddot{\chi} + \omega(x)^2\chi$ in equation (27) is given by

$$\chi \propto \omega(x)^{-\frac{1}{2}} exp(\pm i\omega(x)x), \tag{28}$$

and is a good approximation whenever the time variation $\omega(x)$ over one period is negligible. Higher order WKB solutions include higher order derivatives of $\omega(x)$.

On sub-horizon scales ($x \gg x_*$), the mode χ in equation (27) oscillates ($\omega \simeq k$), which implies $Q \simeq 0$ and the standard WKB approximation can be applied without difficulty since the condition in equation (24) is satisfied. On super-horizon scales ($x \ll x_*$), the perturbations exponentially decay or grow and the standard WKB approximation can again be applied. One then matches the two approximate solutions at $x = x_*$ to obtain the super-horizon amplitudes at $x \ll x_*$ which determine the CMB spectra. This procedure opened up the possibility of further improving the knowledge of the spectra by including subsequent adiabatic orders. However, a straightforward generalization to higher adiabatic orders is seriously hindered by the lack of a general prescription for matching the two WKB branches with sufficient accuracy due to the divergence of the quantity Q at the turning point.

This problem was discussed in [20], where it was suggested to replace the standard (plane wave-like) WKB functions with Bessel's functions which are able to give a good approximation at the turning point, whereas the standard WKB approximation require the matching with yet another particular solution at some (unspecified) points both to the left and to the right of the turning point.

An improved WKB analysis of cosmological perturbations was then presented by [20], in order to study cosmological perturbations beyond the lowest order based on Bessel's functions that approximate the true perturbation modes over the complete range of the independent (Langer) variable, from sub-horizon scale, and include the region near the turning point by employing both a perturbative Green's function technique and an adiabatic (or "semi-classical") expansion (for a linear turning point) in order to compute higher order corrections.

The basic idea was to use approximate expressions which are valid for all the coordinate x and then expand round them by considering all the solutions to equation (27) for $x > x_*$ to be written as a linear combination of two functions as

$$u_{\pm}(x) = \sqrt{\frac{\xi(x)}{\omega(x)}}\, J_{\pm m}[\xi(x)], \tag{29a}$$

where

$$\xi(x) = \int_{x_*}^{x} \omega(y)dy \quad , \quad m = \frac{1}{n+2} \ , \tag{29b}$$

and J_v are Bessels functions. For a general frequency ω, equation (29a) satisfy

$$\left[\frac{d^2}{dx^2} + \text{w}^2(x) - \sigma(x)\right] u_{\pm}(x) = 0, \tag{30}$$

where the quantity

$$\sigma(x) = \frac{3}{4}\frac{(\omega')^2}{\omega^2} - \frac{\omega''}{2\omega} + \left(m^2 - \frac{1}{4}\right)\frac{\text{w}^2}{\xi^2} = Q + \left(m^2 - \frac{1}{4}\right)\frac{\text{w}^2}{\xi^2}, \tag{31}$$

contains the term Q (primes denote derivatives with respect to the argument of the given function) as defined in equation (23), whose divergent behavior at the turning point $x = x_*$was the possible cause of failure of the standard WKB approximation.

For a general (finite) frequency which can be expanded in powers of $x = x_*$ in the form

$$\omega^2 = C(x - x_*)^n\left[1 + \sum_{q\geq 1} c_q (x - x_*)^n\right], \tag{32}$$

it was found that the extra term in the quantity $\sigma(x)$ as defined in equation (31) precisely "removed" the divergence of Q at the turning point $x = x_*$, and the WKB method could therefore be extended to higher orders. In fact, the residue

$$\lim_{x\to x_*}\sigma(x) = \frac{3(n+5)c_1^2}{2(n+4)(n+6)} - \frac{3c_2}{n+6}, \tag{33}$$

is finite. The finiteness of $\sigma(x)$ at the turning point was crucial in order to extend the WKB method beyond the leading order.

In the improved WKB analysis, the asymptotic expression of the mode function were obtained taking into account the asymptotic form of the Bessel functions J_v for $x \to \infty$ in the form

$$J_{\pm m}[\xi_1(x)] \sim \sqrt{\frac{2}{\pi\xi_1(x)}} cos\left(\xi_1(x) \mp \frac{\pi}{2}m - \frac{\pi}{4}\right), \tag{34}$$

and therefore, gave general formulae for the amplitudes, the spectral indices, and the "runnings" of these fluctuations next-to-leading order both in the adiabatic expansion previously done by R. E Langer, and a new perturbative expansion which makes use of the Green's function technique.

2.6. Factorization and Boost Transformation procedures

A procedure for obtaining progressively improving approximate solutions of the WKB (semi-classical) model of the stationary Schrödinger equation was presented by [17]. If $k^2 = constant$, then the equation of dynamics governing the evolution of cosmological perturbations in single field inflationary scenario given in equation (2) is similar to the time-independent Schrödinger equation that is normally dealt with in quantum mechanics when reorganized to take the form

$$\left[\frac{d^2}{d\eta^2} + \omega^2(k,\eta)\right] \square(\eta) = k^2 \square(\eta). \tag{35}$$

In his work, each order of approximation provides an amplitude modulated "plane" wave function specified by a renormalized momentum. A simple binomial expansion of the renormalized momentum allows exact evaluation of the phase accumulation integral for studying basic features of the dynamics in arbitrary potentials whereas for a linear potential, the probability density profile reveals the expected confinement of the particle within the allowed region. The method is more elegant compared to perturbation and other expansion methods which are generally tedious and the general solutions apply to all types of second-order differential equations similar to the Schrodinger equation, which are generally used in Cosmology, Mathematics, Physics, Chemistry, Biology, Economics, and other disciplines where such second-order processes occur.

This study applies a factorization and successive boost transformation procedures, a method used in quantum mechanics in cosmology to study cosmological perturbations during inflation, and the results obtained are compared with those obtained through the standard WKB approximation.

CHAPTER THREE

METHODOLOGY

3.1 Factorization

The form of the second-order differential operator $(\frac{d^2}{d\eta^2} + \omega^2(k,\eta))$ in equation (2), which noting the successive operation

$$\frac{d^2}{d\eta^2}\square = \frac{d}{d\eta}\frac{d}{d\eta}\square = \left(\frac{d}{d\eta}\right)^2\square\,, \tag{36a}$$

re-written in the form

$$\frac{d^2}{d\eta^2} + \omega^2(k,\eta)) \equiv \left(\frac{d}{d\eta}\right)^2 + \omega^2(k,\eta)\ , \tag{36b}$$

is expressible as a difference of two squares after introducing the imaginary number $i = \sqrt{-1}$ according to

$$a^2 + b^2 = a^2 - (ib)^2, \tag{36c}$$

which can be easily factorized depending on the forms of a and b. Applying this in equation (2) and taking into account of the effective time-dependent frequencies given in equation (3) provides two alternative factorised forms

$$\left(-i\frac{d}{d\eta} + \omega\right)\left(i\frac{d}{d\eta} + \omega\right)\square = -i\frac{d\omega}{d\eta}\square \quad ,` \tag{37a}$$

$$\left(i\frac{d}{d\eta} + \omega\right)\left(-i\frac{d}{d\eta} + \omega\right)\square = i\frac{d\omega}{d\eta}\square, \tag{37b}$$

according to the ordering of the differential operators, noting that the term $\pm i \frac{d}{d\eta} \square$, arises from the application of the factorized differential operator on $\square$ to obtain the original form in equation (2).

Introducing complex mode functions ϕ and ϕ^* defined by

$$\phi = \left(i \frac{d}{d\eta} + \omega\right) \square = i \frac{d\square}{d\eta} + \omega\, \square \quad ; \quad \phi^* = \left(-i \frac{d}{d\eta} + \omega\right) \square = -i \frac{d\square}{d\eta} + \omega\, \square\,, \tag{38a}$$

with

$$\square = \frac{1}{2\omega}(\phi + \phi^*) \quad ; \quad \frac{d\square}{d\eta} = \frac{1}{2i}(\phi - \phi^*), \tag{38b}$$

puts equation (37a) – (37b) in the simpler first-order form

$$i \frac{d\phi}{d\eta} = \left(\omega + \frac{i}{2\omega} \frac{d\omega}{d\eta}\right) \phi + \frac{i}{2\omega} \frac{d\omega}{d\eta} \phi^*, \tag{39a}$$

$$i \frac{d\phi^*}{d\eta} = \left(-\omega + \frac{i}{2\omega} \frac{d\omega}{d\eta}\right) \phi^* + \frac{i}{2\omega} \frac{d\omega}{d\eta} \phi. \tag{39b}$$

Introducing reduced mode functions $\bar{\phi}$ and $\bar{\phi}^*$ defined by

$$\phi = \sqrt{\omega}\bar{\phi} \quad ; \quad \phi^* = \sqrt{\omega}\bar{\phi}^* \quad ; \quad \square(\eta) = \frac{1}{2\sqrt{\omega}}\left(\bar{\phi} + \bar{\phi}^*\right) \quad ; \quad \frac{d\square}{d\eta} = \frac{\sqrt{\omega}}{2i}\left(\bar{\phi} - \bar{\phi}^*\right), \tag{40a}$$

with

$$\frac{d\phi}{d\eta} = \frac{1}{2\sqrt{\omega}} \frac{d\omega}{d\eta} \bar{\phi} + \sqrt{\omega} \frac{d\bar{\phi}}{d\eta} \quad ; \quad \frac{d\phi^*}{d\eta} = \frac{1}{2\sqrt{\omega}} \frac{d\omega}{d\eta} \bar{\phi}^* + \sqrt{\omega} \frac{d\bar{\phi}^*}{d\eta}\ , \tag{40b}$$

reduces equations (39a) - (39b) to the much simpler and convenient first-order forms

$$i\frac{d\bar{\phi}}{d\eta} = \omega\,\bar{\phi} + \frac{i}{2\omega}\frac{d\omega}{d\eta}\bar{\phi}^*, \tag{41a}$$

$$i\frac{d\bar{\phi}^*}{d\eta} = -\omega\,\bar{\phi}^* + \frac{i}{2\omega}\frac{d\omega}{d\eta}\bar{\phi}. \tag{41b}$$

These are coupled equations for the mode functions ϕ and ϕ^*. Equations (41a) and (41b), together with the definition of the effective time-dependent frequencies in equation (3), and the mode function $\square(\eta)$ in equation (40a), are the final forms of the factorized mode equation governing the evolution of cosmological perturbations in a single field scenario given in equation (2). An important point to note is that for real mode function $\square(\eta)$, the mode functions ϕ and ϕ^* as defined in equation (38a) are complex conjugates while for complex mode function $\square(\eta)$, the mode functions ϕ and ϕ^* would not be related by simple complex conjugation. However, equations (41a) - (41b) apply for real or complex mode function $\square(\eta)$ since they follow from the operator ordering in the factorization and definitions given in equation (37a) - (37b) and (38a) respectively. We may then adopt a general notation

$$\phi \rightarrow \phi_- \quad ; \quad \phi^* \rightarrow \phi_+ \quad ; \; \bar{\phi} \rightarrow \bar{\phi}_- \, ; \; \bar{\phi}^* \rightarrow \bar{\phi}^*{}_+ \quad ; \; \square(\eta) = \frac{1}{2\sqrt{\omega}}\left(\bar{\phi}_- + \bar{\phi}^*{}_+\right), \tag{42}$$

defined according to equation (38a), applying to real $\square(\phi_+ = \phi_-^*)$ or complex $\square\,(\phi_+ \neq \phi_-^*)$.

3.2 The matrix form of factorization

Equation (41a) and (41b) can be written into a matrix equation as

$$i\frac{dX}{d\eta} = HX, \tag{43a}$$

after introducing a two-component column matrix $\mathrm{X}(\eta)$ defined by

$$X(\eta) = \begin{pmatrix} \bar{\Phi} \\ \bar{\Phi}^{*} \end{pmatrix}, \tag{43b}$$

and H is the corresponding Hamiltonian obtained as

$$H = \begin{pmatrix} \omega & \frac{i}{2\omega}\frac{d\omega}{d\eta} \\ \frac{i}{2\omega}\frac{d\omega}{d\eta} & -\omega \end{pmatrix}. \tag{43c}$$

Introducing Pauli matrices σ_x , σ_y, σ_z, and the identity matrix I defined by,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;;\; \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;;\; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{43d}$$

the Hamiltonian in equation (43c) is expressed in the form

$$H = \omega\sigma_z + if\sigma_x, \tag{43e}$$

where we identify a factorization coupling parameter $f(\eta)$ defined by

$$f(\eta) = \frac{1}{2\omega}\frac{d\omega}{d\eta}. \tag{43f}$$

The time-dependence of the coefficients ω and $f(\eta)$ of the Hamiltonian H in equation (43e) does not allow exact analytical solution of the matrix equation (43a) through direct integration due to the non-commutativity of σ_z and σ_x . We therefore, need to transform the original equation of dynamics in equation (43a), with Hamiltonian in equation (43e) to a dynamical frame where the resultant Hamiltonian H is either time-independent for an exact analytical solution or has a negligible approximation parameter for high accuracy approximation.

3.3. The transformation law

We develop the transformation law by considering that the transformation from one dynamical frame to another of higher accuracy level is taken one step at a time. In this respect, we consider the general case of a transformation from the n^{th}-order dynamical frame characterized by the mode function matrix X_n , Hamiltonian H_n, approximation parameter ς_n, and transformation operator T_n specified by arbitrary transformation parameter θ_n, to the $(n+1)^{th}$-order frame. The equation of dynamics in the n^{th}-order dynamical frame takes the form

$$i\frac{dX_n}{d\eta} = H_n X_n \ , n = 0,1,2, \dots \dots \dots \tag{44a}$$

Transformation to the $(n+1)^{th}$-order dynamical frame characterized by mode function X_{n+1} and transformation operator T_{n+1} is defined by

$$X_{n+1} = T_{n+1} X_n \ \Rightarrow X_n = T_{n+1}^{-1} X_{n+1} \quad ; \quad \frac{dX_n}{d\eta} = T_{n+1}^{-1}\frac{dX_{n+1}}{d\eta} + X_{n+1}\frac{dT_{n+1}^{-1}}{d\eta} . \tag{44b}$$

In general, a transformation operator T and its inverse T^{-1} satisfy the condition

$$TT^{-1} = T^{-1}T = I, \tag{44c}$$

where I is the identity matrix.

We substitute $X_n = T_{n+1}^{-1} X_{n+1}$ from equation (44b) into equation (44a), multiply the result by T_{n+1} from the left and apply the condition (44c) to obtain the equation of dynamics in the $(n+1)^{th}$-order dynamical frame in the form

$$i \frac{dX_{n+1}}{d\eta} = H_{n+1} X_{n+1} \, , n = 0,1,2, \dots \dots \dots \, , \tag{44d}$$

where the Hamiltonian H_{n+1} is obtained as a transformation of H_n according to

$$H_{n+1} = T_{n+1} H_n T_{n+1}^{-1} - i T_{n+1} \frac{dT_{n+1}^{-1}}{d\eta}. \tag{44e}$$

3.4. The boost frames

According to equation (43e), the original Hamiltonian H is non-Hermitian. We therefore consider that the appropriate transformation to apply is a boost along an axis normal to the plane of the Hamiltonian. The zeroth-order dynamical frame Hamiltonian $H_o = H$ given in equation (43e) is expressed as

$$H_o = q_o(k,\eta)\sigma_z \, + \, i\varsigma_o \sigma_x \, , \tag{45a}$$

where $q_o(k,\eta)$ and ς_o are the zeroth-order renormalized time-dependent frequency and dynamical approximation parameter, respectively defined by comparing equation (43e) and (45a) in the form

$$q_o(k,\eta) = \mathrm{w}(k,\eta) \quad ; \quad \varsigma_o = f = \frac{1}{2\mathrm{w}} \frac{d\mathrm{w}}{d\eta} \, . \tag{45b}$$

The fact that $H_o = H$ is defined in the zx-plane means that the original (zeroth-order) dynamical frame of the mode equation (2) governing the evolution of cosmological perturbations in a single field inflationary scenario is the zx-plane. The first-order boost operator $T_1(\eta)$ is therefore defined along the $y-axis$ specified by an arbitrary transformation parameter $\theta_1(\eta)$ according to the definition

$$T_1(\eta) = e^{\frac{1}{2}\theta_1(\eta)\sigma_y} = \begin{pmatrix} cosh\frac{1}{2}\theta_1 & -\mathrm{isinh}\frac{1}{2}\theta_1 \\ \mathrm{isinh}\frac{1}{2}\theta_1 & cosh\frac{1}{2}\theta_1 \end{pmatrix}, \tag{45c}$$

which on substituting into equation (44e) for $n = 0$, $H_o = H$ and then eliminating the arbitrary $\theta_1(\eta)$ as explained below provides the first-order dynamical frame Hamiltonian H_1 in the form

$$H_1 = q_1(k,\eta)\sigma_z + i\varsigma_1 \sigma_y , \tag{45d}$$

where $q_1(k,\eta)$ and ς_1 are the first-order renormalized time-dependent frequency and dynamical approximation parameter, respectively. We observe that H_1 is defined in the $zy-plane$, meaning that the first-order dynamical frame is the $zy-plane$. The transformation from the fist-order dynamical frame is therefore, a boost along the $x-axis$.

In general, the accuracy level dynamical frames alternate between zx and zy planes so that the corresponding boost transformation along axes normal to the dynamical planes are affected by boost transformation defined alternately along $y-axis$ and $x-axis$ as appropriate. For the general transformation from the n^{th}-order frame to the $(n+1)^{th}$-order specified above, we define $T_{n+1}(\eta)$ and it's inverse $T_{n+1}^{-1}(\eta)$ in relation to the plane H_n according to

$$H_n = q_n(k,\eta)\sigma_z - i\varsigma_n\sigma_x \Longrightarrow T_{n+1}(\eta) = e^{\frac{1}{2}\theta_{n+1}(\eta)\sigma_y} = \begin{pmatrix} cosh\frac{1}{2}\theta_{n+1} & -isinh\frac{1}{2}\theta_{n+1} \\ isinh\frac{1}{2}\theta_{n+1} & cosh\frac{1}{2}\theta_{n+1} \end{pmatrix} \;\; ;$$

$$T_{n+1}^{-1}(\eta) = e^{-\frac{1}{2}\theta_{n+1}(\eta)\sigma_y} = \begin{pmatrix} cosh\frac{1}{2}\theta_{n+1} & isinh\frac{1}{2}\theta_{n+1} \\ -isinh\frac{1}{2}\theta_{n+1} & cosh\frac{1}{2}\theta_{n+1} \end{pmatrix}, \tag{46a}$$

which we substitute into the transformation law in equation (44e) to obtain

$$H_{n+1} = q_n\{(cos\text{h}\theta_{n+1} + \bar{\varsigma}_n sin\text{h}\theta_{n+1})\delta_z + i(sin\text{h}\theta_n + \bar{\varsigma}_n cos\text{h}\theta_{n+1})\delta_x\} + i\frac{1}{2}\frac{d\theta_{n+1}(\eta)}{d\eta}\sigma_y. \quad (46b)$$

We have used standard hyperbolic function identities

$$cos\text{h}\theta = sin\text{h}^2\frac{1}{2}\theta + cos\text{h}^2\frac{1}{2}\theta \quad ; \quad sin\text{h}\theta = 2cos\text{h}\frac{1}{2}\theta sin\text{h}\frac{1}{2}\theta, \quad (46c)$$

to obtain the final form in equation (46b), and introduced a parameter $\bar{\varsigma}_n$(short form for factorization approximation parameter) defined by

$$\bar{\varsigma}_n(\eta) = \frac{\varsigma_n}{q_n}. \quad (46d)$$

3.5. Eliminating $\boldsymbol{\theta_{n+1}(\eta)}$: renormalized time-dependent frequency

The only externally introduced parameter in H_{n+1} in equation (46b) is the boost transformation parameter $\theta_{n+1}(\eta)$ which must now be eliminated to obtain a physically meaningful dynamical frame Hamiltonian H_{n+1}. Since we shall diagonalize H_{n+1} in the end to obtain the desired approximate solution of equation (44d) in the $(n+1)^{th}$-order dynamical frame. We start by eliminating the σ_x -component in equation (46b), by setting the co-efficient to zero, that is to say, the vanishing of the off diagonal elements implies that

$$sin\text{h}\theta_{n+1} + \bar{\varsigma}_n cos\text{h}\theta_{n+1} = 0\,, \quad (47a)$$

which easily fixes the boost parameter θ_{n+1} in terms of the physical parameters in the form

$$tan\text{h}\theta_{n+1} = -\,\bar{\varsigma}_n. \quad (47b)$$

The boost transformation parameter is defined in terms of the factorization approximation parameter depending only on the time dependent frequency $\omega(k,\eta)$ given by the expression in equation (3)

Squaring equation (47b), using, $cosh^2\theta - sinh^2\theta = 1$, $tanh^2\theta = \frac{sinh^2\theta}{cosh^2\theta}$, gives

$$cosh\theta_{n+1}(\eta) = \frac{1}{\sqrt{1-\bar{\varsigma}_n^2(\eta)}} \qquad ; \quad sinh\theta_{n+1}(\eta) = -\frac{\bar{\varsigma}_n(\eta)}{\sqrt{1-\bar{\varsigma}_n^2(\eta)}}. \tag{47c}$$

where the negative sign in the definition of $tanh\theta_{n+1}$ has been taken into account in the derivation of $cosh\theta_{n+1}(\eta)$ and $sinh\theta_{n+1}(\eta)$.

Using equation (47c), the co-efficient of δ_z in equation (46) is obtained in the form

$$cosh\theta_{n+1} + \bar{\varsigma}_n sinh\theta_{n+1} = \sqrt{1-\bar{\varsigma}_n^2(\eta)}. \tag{47d}$$

Differentiating equation (47b) with respect to η using

$$\frac{d}{d\eta} tanh\theta_{n+1} = \frac{d\theta_{n+1}}{d\eta}\frac{d}{d\theta_{n+1}} tanh\theta_{n+1} = \frac{d\theta_{n+1}}{d\eta}(1 - tanh^2\theta_{n+1}) \, , \tag{47e}$$

gives the final result

$$\frac{1}{2}\frac{d\theta_{n+1}}{d\eta} = -\frac{1}{2\left(1-\bar{\varsigma}_n^2(\eta)\right)}\frac{d\bar{\varsigma}_n(\eta)}{d\eta}. \tag{47f}$$

Noticing that the right hand side of equation (47f) involves first-order derivative of the n-order approximation parameter $\bar{\varsigma}_n(\eta) = \frac{\varsigma_n}{q_n}$, we introduce the $(n+1)^{th}$ boost approximation parameter $\varsigma_{n+1(\eta)}$ defined by

$$\frac{1}{2}\frac{d\theta_{n+1}}{d\eta} = -\varsigma_{n+1}(\eta) \Longrightarrow \varsigma_{n+1}(\eta) = \frac{1}{2\left(1-\bar{\varsigma}_n^2(\eta)\right)}\frac{d\bar{\varsigma}_n(\eta)}{d\eta} \quad ; \quad n = 0,1,2,3, \dots \dots , \tag{47g}$$

Substituting equations (47a), (47d) and (47g) into equation (46b) (noting that the coefficients are the same), we obtain the boost frame Hamiltonian H in the alternate form

$$H_n = q_n\sigma_z - i\varsigma_n\sigma_x \Longrightarrow H_{n+1} = q_{n+1}\sigma_z - i\varsigma_{n+1}\sigma_y \qquad , \qquad (48a)$$

where we have introduced the $(n+1)^{th}$-order renormalized time-dependent frequency obtained as

$$q_{n+1}(\eta) = q_n(\eta)\sqrt{1-\bar{\varsigma}_n^2(\eta)} \quad ; \qquad n = 0, 1, 2, 3, \dots \dots, \qquad (48b)$$

Setting $n = 0, 2, 3, \dots \dots,$ in equation (48a) - (48b) provides the zeroth, even, and odd order dynamical frame Hamiltonians H according to

$$H_o = q_o\sigma_z - i\varsigma_o\sigma_x \quad ; \quad q_o = \omega \quad ; \quad \varsigma_o = \frac{1}{2\omega}\frac{d\omega}{d\eta} \quad ; \quad \bar{\varsigma}_o = \frac{\varsigma_o}{q_o} = \frac{1}{2\omega^2}\frac{d\omega}{d\eta} \ , \qquad (48c)$$

$$H_{2n} = q_{2n}\sigma_z - i\varsigma_{2n}\sigma_x \quad ; \quad q_{2n} = q_{2n-1}\sqrt{1-\bar{\varsigma}_{2n-1}^2} \quad ; \quad \varsigma_{2n} = \frac{1}{2\left(1-\bar{\varsigma}_n{}^2(\eta)\right)}\frac{d\bar{\varsigma}_n(\eta)}{d\eta} \quad ;$$
$$\bar{\varsigma}_{2n-1} = \frac{\varsigma_{2n-1}}{q_{2n-1}} \quad , \ n \geq 1, \qquad (48d)$$

$$H_{2n+1} = q_{2n+1}\sigma_z - i\varsigma_{2n+1}\sigma_y \quad ; \quad q_{2n+1} = q_{2n}\sqrt{1-\bar{\varsigma}_{2n-1}^2} \quad ; \quad \varsigma_{2n+1} = \frac{1}{2\left(1-\bar{\varsigma}_{2n}^2\right)}\frac{d\bar{\varsigma}_{2n}(\eta)}{d\eta} \quad ;$$
$$\bar{\varsigma}_{2n} = \frac{\varsigma_{2n}}{q_{2n}} \quad , \ n \geq 0, \qquad (48e)$$

where $\bar{\varsigma}_o$ is the short form for zeroth-order factorization in equation (48c) approximation parameter as defined in equation (46d) and given in equation (48c).

3.6. Diagonalization and Approximate Solutions

We recall that the purpose of the transformation of the original equation (43a) and its Hamiltonian in equation (43e) is to find a dynamical frame in which the resultant Hamiltonian H is either time-independent for an exact analytical solution or has a negligible approximation parameter for a high accuracy approximation. Since the renormalized frequency $q_n(\eta)$ and the dynamical approximation parameter $\varsigma_n(\eta)$ which specify the Hamiltonian H_n in the n^{th}-order dynamical frame for $n = 0, 1, 2, \ldots$, are time-dependent, the resultant equation of dynamics (44a) can only be solved under good approximation conditions in which we consider the n^{th}-order dynamical approximation parameter $\varsigma_n(\eta)$ to be negligible. Under such conditions, Hamiltonian H_n is diagonalized, leading to a simple solution through direct integration.

The main task is to establish that the n^{th}-order dynamical approximation parameter $\varsigma_n(\eta)$ is small enough to be neglected at the $n \geq 0$ accuracy level. To do this, we consider that $\varsigma_n(\eta)$ is defined in terms of progressively increasing orders of differentiation of the zeroth-order approximation parameter $\varsigma_o(\eta)$.

For $n = 0$, the zeroth-order approximation parameter takes the form

$$\varsigma_o = \frac{1}{2\text{ш}}\frac{d\text{ш}}{d\eta} \qquad ; \qquad \bar{\varsigma}_o = \frac{\varsigma_o}{q_o} = \frac{1}{2\text{ш}^2}\frac{d\text{ш}}{d\eta}\,. \tag{49}$$

The factorization approximation parameter for $n \geq 1$ is given by

$$n \geq 1 \;\; : \quad \varsigma_n = \frac{1}{2\left(1-\bar{\varsigma}_{n-1}^2\right)}\frac{d\bar{\varsigma}_{n-1}}{d\eta} \qquad ; \qquad \bar{\varsigma}_{n-1} = \frac{\varsigma_{n-1}}{q_{n-1}}. \tag{50a}$$

For first order approximation we set $n = 1$, and on using equation (49), we obtain the first order factorization approximation parameter in the form

$$n = 1 \quad : \qquad \varsigma_1 = \frac{1}{2(1-\bar{\varsigma}_o^2)}\frac{d\bar{\varsigma}_o}{d\eta} = \frac{1}{4\left(1-\left(\frac{1}{2w^2}\frac{dw}{d\eta}\right)^2\right)}\frac{d}{d\eta}\left(\frac{1}{w^2}\frac{dw}{d\eta}\right). \tag{50b}$$

We use equation (50b) to obtain

$$\varsigma_1 = \frac{1}{2(1-\bar{\varsigma}_o^2)}\frac{d\bar{\varsigma}_o}{d\eta} = \frac{1}{2\left(1-\left(\frac{1}{2w^2}\frac{dw}{d\eta}\right)^2\right)}\left(\frac{1}{\omega}\left(\frac{\omega''}{2\omega} - \frac{(\omega')^2}{\omega^2}\right)\right). \tag{50c}$$

The first-order dynamical frame in equation (50b) is characterized by the first-order approximation parameter ς_1 obtained in terms of the first-order derivatives of the zeroth-order approximation parameter $\bar{\varsigma}_0$.

The derivative of the zeroth-order approximation parameter with respect to η is obtained as

$$\frac{d\bar{\varsigma}_o}{d\eta} = \frac{d}{d\eta}\left(\frac{1}{2\omega^2}\frac{d\omega}{d\eta}\right) = \frac{\omega''}{2\omega^2} - \frac{(\omega')^2}{\omega^3} = \frac{1}{\omega}\left(\frac{\omega''}{2\omega} - \frac{(\omega')^2}{\omega^2}\right). \tag{50d}$$

Equation (50d) produces an expression that is exact and similar to the quantity $Q(k,\eta)$ used in the standard WKB approximation condition in equation (23), especially the expression $\frac{\omega''}{2\omega} - \frac{(\omega')^2}{\omega^2}$, though with different coefficients

$$Q = -\left(\frac{\omega''}{2\omega} - \frac{3}{4}\frac{(\omega')^2}{\omega^2}\right) = -\left(\frac{\omega''}{2\omega} - \frac{(\omega')^2}{\omega^2} + \frac{1}{4}\frac{(\omega')^2}{\omega^2}\right) = -\left(\omega\frac{d\bar{\varsigma}_o}{d\eta} + \frac{1}{4}\frac{(\omega')^2}{\omega^2}\right) \Longrightarrow \left|\frac{d\bar{\varsigma}_o}{d\eta}\right| = \left|\frac{1}{\omega}\left(Q + \frac{1}{4}\frac{(\omega')^2}{\omega^2}\right)\right|. \tag{50e}$$

Equation (50d), obtained through boost transformation and diagonalization procedure is more exact and elegant compared to the quantity $Q(k,\eta)$ used in the standard WKB analysis of cosmological perturbations obtained by first assuming a solution to the mode equation (2) in the form of an ansatz (trial solution) given in equation (21), and whose divergent behaviour at the turning point was identified as the possible cause of failure of the standard WKB approximation.

The second-order dynamical frame will be characterized by the second-order approximation parameter ς_2 obtained in terms of the first-order derivatives of the first-order approximation parameter $\bar{\varsigma}_1$. In general, the n^{th}-order dynamical frame characterized by the n^{th}-order approximation parameter ς_n, is obtained in terms of the first-order derivatives of the $(n-1)^{th}$ order approximation parameter $\bar{\varsigma}_{n-1}$, as given in equation (50a), such that $\varsigma_n \ll \varsigma_o$.

The dynamical factorization approximation parameter therefore becomes progressively smaller with increasing $n \geq 0$. This means that the accuracy level increases with the number $n \geq 0$ of successive transformations from the zeroth-order to the n^{th}-order $(n \geq 1)$ dynamical frame. The highest accuracy level is achieved in the dynamical frame where the approximation parameter $\varsigma(\eta)$ takes the smallest possible value and can be safely neglected.

Hence for n^{th}-level accuracy (equivalent to approximation to the n^{th} $-$order), we set $\varsigma_n(\eta)$ (general $n = 0$, even, odd) equal to zero in any of the forms in equations in (48a) - (48e), leading to diagonalization of the general n^{th} $-$order dynamical Hamiltonian according to

$$\varsigma_o(\eta) = 0 \quad \Longrightarrow H_o = q_o\sigma_z \quad , \quad q_o(\eta) = \omega(\eta) \quad ; \quad \bar{\varsigma}_o = \frac{1}{2\omega^2}\frac{d\omega}{d\eta} , \qquad (51a)$$

$$\varsigma_n(\eta) = 0 \quad \Longrightarrow \quad H_n = q_n\sigma_z \quad , \quad q_n(\eta) = q_{n-1}(\eta)\sqrt{1 - \bar{\varsigma}^2_{2n-1}(\eta)} \quad , \; n = 1,2,3..,. \qquad (51b)$$

Substituting the diagonalized Hamiltonian from equations (51a) - (51b) into equation (44a), we easily obtain approximate solution satisfying the accuracy conditions (51a) - (51b) in the n^{th} $-$order dynamical frame in the form

$$X_n(\eta) = \mho_n(\eta) X_n(0) \quad , \quad \mathrm{n} = 0, 1, 2, \ldots, \tag{51c}$$

with the time evolution operator $\mho_n(\eta)$ obtained through direct integration in the final form

$$\mho_n(\eta) = e^{-i\delta_n(\eta)\sigma_z} \quad , \quad n = 0,1,2,3 \ldots \ldots, \tag{51d}$$

where $\delta_n(\eta)$ is the phase accumulation integral obtained in the form

$$\delta_n(\eta) = \int_{\eta_o}^{\eta} q_n(\eta^{'}) d\eta^{'} \quad , \quad n = 0,1,2,3 \ldots \ldots,. \tag{51e}$$

Noting that $X_n(\eta)$ in the n^{th} $-$order dynamical frame is obtained from $X(\eta)$ as defined in equation (43b) in the original frame through a succession of boost transformations according to

$$X_n(\eta) = \mathrm{T_n}(\eta) X_{n-1}(\eta) = \mathrm{T_n}(\eta)\mathrm{T_{n-1}}(\eta)\mathrm{T_{n-2}}(\eta) \ldots \ldots \mathrm{T_3}(\eta)\mathrm{T_2}(\eta)\mathrm{T_1}(\eta)\mathrm{T_o}(\eta) X(\eta) \ , \tag{52a}$$

where

$$\mathrm{T_o}(\eta) = \mathrm{I} \ \Longrightarrow \ X_o(\eta) = \mathrm{T_o}(\eta) X(\eta) = X(\eta). \tag{52b}$$

We apply inverse operations in succession from the left of equation (52a), starting with $T_n^{-1}(\eta)$ as appropriate, to obtain the original mode amplitude in the form ($T_0^{-1}(\eta) = \mathrm{I}$)

$$X(\eta) = T_1^{-1}(\eta) T_2^{-1}(\eta) T_3^{-1}(\eta) \ldots \ldots T_{n-2}^{-1}(\eta) T_{n-1}^{-1}(\eta) T_n^{-1}(\eta) X_n(\eta). \tag{52c}$$

Applying the inverse operation on equation (51c) from the left and substituting equation (52c), together with the entry-boundary transformation

$$X_n(0) = \mathrm{T_n}(0)\mathrm{T_{n-1}}(0)\mathrm{T_{n-2}}(0) \ldots \ldots \mathrm{T_3}(0)\mathrm{T_2}(0)\mathrm{T_1}(0) X(0), \tag{52d}$$

we obtain the approximate solution of equation (43a) in the original frame in the form

$$X(\eta) = \mho(\eta)X(0) \qquad , \qquad X(\eta) = \begin{pmatrix} \overline{\Phi}\,(\eta) \\ \overline{\Phi}^{*}(\eta) \end{pmatrix} \qquad , \qquad X(0) = \begin{pmatrix} \overline{\Phi}\,(0) \\ \overline{\Phi}^{*}(0) \end{pmatrix} \,, \tag{52e}$$

where the time evolution operator $\mho(\eta)$ in the original frame has been obtained in the form $\mho(\eta) = T_1^{-1}(\eta)T_2^{-1}(\eta) \ldots T_{n-1}^{-1}(\eta)T_n^{-1}(\eta)\mho_n(\eta)\mathrm{T_n}(0)\mathrm{T_{n-1}}(0) \ldots \mathrm{T_2}(0)\mathrm{T_1}(0) \;\; , \; n \geq 1.$

(52f)

We recall that the boost operators applied in succession in equations (52a) - (52f) alternate between the $x -$ axis and $y -$ axis boosts as explained earlier in the form

$$\mathrm{T_{2j}}(\eta) = e^{\frac{1}{2}\theta_{2j}(\eta)\sigma_x} \quad ; \quad \mathrm{T_{2j+1}}(\eta) = e^{\frac{1}{2}\theta_{2j+1}(\eta)\sigma_y} \quad , \quad j = 0,1,2,3, \ldots, \; ; \; \theta_0(\eta) = 0. \tag{52g}$$

Once $\mho(\eta)$ is evaluated explicitly and substituted into equation (52e) to obtain $\overline{\Phi}(\eta)$ and $\overline{\Phi}^{*}(\eta)$, the desired general solution of the equation of dynamics governing the evolution of cosmological perturbations in single field inflationary scenario in equation (2) up to the $n^{th} -$ order accuracy follows easily using the definition of the mode function ⍰(η) in equation (40a). We illustrate the procedure by expressing explicit results for the zeroth-order in chapter four.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1. Zeroth - Order Approximation

Up to zeroth-order approximation, we set

$$n = 0 \quad : \quad q_o(\eta) = \omega(\eta) \qquad ; \qquad \delta_0(\eta) = \int_{\eta_o}^{\eta} q_0(\eta^{'})d\eta^{'}, \tag{53a}$$

$$\mho_0(\eta) = e^{-i\delta_0(\eta)\sigma_z} = \begin{pmatrix} e^{-i\frac{\delta_0}{2}(\eta)} & 0 \\ 0 & e^{i\frac{\delta_0}{2}(\eta)} \end{pmatrix}, \tag{53b}$$

$$T_0^{-1}(\eta) = \mathrm{I} = \mathrm{T_o}(\eta), \tag{53c}$$

to obtain

$$\mho(\eta) = \mho_0(\eta) \qquad \Longrightarrow \mho(\eta) = \begin{pmatrix} e^{-i\frac{\delta_0}{2}(\eta)} & 0 \\ 0 & e^{i\frac{\delta_0}{2}(\eta)} \end{pmatrix}. \tag{53d}$$

Substituting equation (53d) into equation (52e) we obtain the final result

$$\overline{\Phi}(\eta) = e^{-i\frac{\delta_0}{2}(\eta)}\overline{\Phi}(0) \qquad ; \qquad \overline{\Phi}^*(\eta) = e^{i\frac{\delta_0}{2}(\eta)}\overline{\Phi}^*(0). \tag{53e}$$

Expressing the entry-boundary complex mode amplitudes $\overline{\Phi}(\eta)$, $\overline{\Phi}^*(\eta)$ in the polar form

$$\overline{\Phi}(0) = \left|\overline{\Phi}(0)\right|e^{-i\vartheta} \qquad ; \qquad \overline{\Phi}^*(0) = \left|\overline{\Phi}(0)\right|e^{i\vartheta} \;\; ; \; \vartheta = \text{constant}\,, \tag{53f}$$

in equation (53e) and using the result in the definition of the mode function $\square(\eta)$ in equation (40a) gives the solution of mode equation (2) to zeroth-order approximation in the form

$$n = 0 \quad : \quad \square(\eta) = \frac{A}{\sqrt{q_0}} cos\left(\int_{\eta_o}^{\eta} q_0(\eta^{'})d\eta^{'} + \vartheta \right) \qquad ; \quad A = \left|\overline{\Phi}(0)\right|, \tag{53g}$$

after substituting $\delta_0(\eta)$ from equation (53a).

We observe that for complex mode function ▯(η), the general definitions in equation (52a) and the results obtained in equation (53e) provide the zeroth-order solution in the form

$$n = 0 : \text{▯}(\eta) = \frac{A}{2\sqrt{q_0}} e^{-i\left(\int_{\eta_o}^{\eta} q_0(\eta')d\eta'\right)} + \frac{B}{2\sqrt{q_0}} e^{i\left(\int_{\eta_o}^{\eta} q_0(\eta')d\eta'\right)} \; ; \; A = \bar{\Phi}(0) \; ; \; B = \bar{\Phi}^{*}(0). \qquad (53h)$$

The zeroth-order solution equation (53h), is exactly the leading order/first- order standard WKB solution, and takes exactly the same form of the assumed solution (ansatz) in the standard WKB approximation in equation (21). Furthermore, it does not require any matching condition about any particular point, that is to say, a turning point as outlined in the results for the WKB mode function and therefore the issue of divergence at a turning point as encountered in the WKB approximation does not arise in the approximation procedure developed in this study.

4.2. Zeroth - Order Power Spectra

Using the expressions for the time dependent frequency in equation (3), the potentials in equation (4a)-(4b), the form of the scale factor $a(\eta)$ in equation (16c)-(16d), and the solution of mode equation (2) to zeroth-order approximation in equation (53g), we develop general expressions for the zeroth-order power spectrum within the factorisation and boost transformation procedure.

4.2.1 Zeroth - Order Power Spectra in a Radiation Dominated Universe

Taking the form of the scale factor for a radiation-dominated universe in equation (16c) as a function of conformal time

$$a(\eta) = \beta\eta \quad ; \quad \beta = \text{constant}, \tag{54a}$$

from which we obtain

$$a'(\eta) = \beta \ , \tag{54b}$$

$$a''(\eta) = 0. \tag{54c}$$

Using the (54a)-(54c) in the definitions of the potentials given in equations (4a)-(4b), we obtain for scalar (density perturbation)

$$Ƶ_S = \sqrt{2}\beta\eta \qquad \Longrightarrow \qquad Ƶ_S'' = 0, \tag{54d}$$

while for tensor (gravitational wave) perturbations

$$Z_T = a(\eta) = \beta\eta \qquad \Longrightarrow \qquad Ƶ_T'' = 0. \tag{54e}$$

Using equations (54d)-(54e) in the definition of the time dependent frequency given by the general expression in equation (3), we obtain expressions for the time dependent frequency for the wave number $k = 1\ and - 1$in the form

$$\omega_S = \sqrt{k^2 - \frac{z_S''}{z_S}} = 1, \tag{54f}$$

$$\omega_T = \sqrt{k^2 - \frac{z_T''}{z_T}} = 1. \tag{54g}$$

For a radiation dominated universe, the form of the scale factor yields similar expressions for the time dependent frequency for scalar perturbations and tensor perturbation as given in equations (54f) and (54g).

Using equations (53a) and equations (54f)-(54g) into the solution for the mode equation obtained in equation (53g), we obtain

$$\mu_{S,T} = Acos\left(\int_0^\eta d\eta + \vartheta\ \right) = Acos(\eta + \vartheta) \quad ; \quad \vartheta = 0, \frac{\pi}{2}, \frac{\pi}{4}\,. \tag{55}$$

Substituting equations (54d) and (55) into the expression for the power spectra in equation (6), we obtain the zeroth-order power spectra in a radiation dominated universe. For scalar perturbations the power spectra is in the form

$$P_\zeta = \frac{1}{8\pi^2}\left|\frac{\mu_S}{z_S}\right|^2 = \frac{A^2 cos^2(\eta+\vartheta)}{16\pi^2\beta^2\eta^2}, \tag{56}$$

while for tensor perturbations is obtained in the form

$$P_h = \frac{2}{\pi^2}\left|\frac{\mu_T}{z_T}\right|^2 = \frac{2A^2 cos^2(\eta+\vartheta)}{\pi^2\beta^2\eta^2}\,, \tag{57}$$

noting that from equations (54f) and (54g), for a radiation-dominated universe $\mu_T = \mu_S$.

For the wave number $k = 0$, the scalar power power spectra takes the form

$$P_\zeta = \frac{A^2 cos^2(\vartheta)}{16\pi^2\beta^2\eta^2}, \tag{58}$$

while the tensor power spectrum has the form

$$P_h = \frac{2A^2 cos^2(\vartheta)}{\pi^2\beta^2\eta^2}. \tag{59}$$

4.2.2. Zeroth - Order Power Spectra in a Matter-Dominated Universe

Taking the form of the scale factor for a matter-dominated universe in equation (16d) as a function of conformal time

$$a(\eta) = \beta\eta^2 \quad ; \quad \beta = \text{constant}, \tag{60a}$$

from which we obtain

$$a'(\eta) = 2\beta\eta\ , \tag{60b}$$

$$a''(\eta) = 2\beta. \tag{60c}$$

Using the (60a)-(60c) in the definitions of the potentials given in equations (4a)-(4b), we obtain for scalar (density perturbation)

$$Ƶ_S = \sqrt{\frac{3}{2}}\beta\eta^2 \qquad \Longrightarrow \qquad Ƶ_S'' = \sqrt{6}\beta, \tag{60d}$$

while for tensor (gravitational wave) perturbations

$$Z_T = a(\eta) = \beta\eta^2 \qquad \Longrightarrow \qquad Ƶ_T'' = 2\beta. \tag{60e}$$

Using equations (60d)-(60e) in the definition of the time dependent frequency given by the general expression in equation (3), we obtain expressions for the time dependent frequency for the wave number $k = 1$ and -1 in the form

$$\omega_S = \sqrt{k^2 - \frac{z_S''}{z_S}} = \sqrt{1 - \frac{2}{\eta^2}}\,, \tag{60f}$$

$$\omega_T = \sqrt{k^2 - \frac{z_T''}{z_T}} = \sqrt{1 - \frac{2}{\eta^2}}\,. \tag{60g}$$

Expressions for the time dependent frequency in equations (60f) and (60g) are also similar for scalar and tensor perturbations respectively.

Using equations (53a) and equations (60f)-(60g) into the solution for the mode equation obtained in equation (53g), we obtain

$$\mu_{S,T} = \frac{A}{\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{4}}} cos\left(\int_0^{\eta} \sqrt{1 - \frac{2}{\eta^2}}\, d\eta + \vartheta \right) \quad ; \quad \vartheta = 0, \frac{\pi}{2}, \frac{\pi}{4}\,. \tag{61}$$

Evaluating $\int \sqrt{1 - \frac{2}{\eta^2}}\, d\eta$, by use of Wolfram Mathematica 8.0 program , we obtain

$$\int \sqrt{1 - \frac{2}{\eta^2}}\, d\eta = \frac{\sqrt{1-\frac{2}{\eta^2}}\,\eta\ \left(\sqrt{-2+\eta^2} + \sqrt{2}\mathrm{ArcTan}\ \left[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}\right]\right)}{\sqrt{-2+\eta^2}}. \tag{62}$$

Using equation (62) in equation (61), we obtain the mode function for the wave number $k = 1\ \ and - 1$ in the form

$$\mu_{S,T} = \frac{A}{\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{4}}} cos\left(\frac{\sqrt{1-\frac{2}{\eta^2}}\,\eta\ \left(\sqrt{-2+\eta^2} + \sqrt{2}ArcTan\ \left[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}\right]\right)}{\sqrt{-2+\eta^2}} + \vartheta\right) \quad ; \quad \vartheta = 0, \frac{\pi}{2}, \frac{\pi}{4}\,. \tag{63}$$

Substituting equations (60d) and (63) into the expression for the power spectra in equation (6), we obtain the zeroth-order power spectra in a matter-dominated universe. For scalar perturbations the power spectra is obtained in the form

$$P_\zeta = \frac{1}{8\pi^2}\left|\frac{\square_S}{z_S}\right|^2 = \frac{A^2}{12\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{2}}} cos^2\left(\frac{\sqrt{1-\frac{2}{\eta^2}}\eta\,(\sqrt{-2+\eta^2}+\sqrt{2}ArcTan\,[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}])}{\sqrt{-2+\eta^2}} + \vartheta\right), \tag{64}$$

while for tensor perturbations it is obtained in the form

$$P_h = \frac{2}{\pi^2}\left|\frac{\square_T}{z_T}\right|^2 = \frac{2A^2}{\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{2}}} cos^2\left(\frac{\sqrt{1-\frac{2}{\eta^2}}\eta\,(\sqrt{-2+\eta^2}+\sqrt{2}ArcTan\,[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}])}{\sqrt{-2+\eta^2}} + \vartheta\right). \tag{65}$$

For the wave number $k = 0$, $\int\sqrt{-\frac{2}{\eta^2}}d\eta$ yields

$$\int\sqrt{-\frac{2}{\eta^2}}d\eta = \sqrt{-\frac{2}{\eta^2}}\,\eta\ \mathrm{Log}[\eta], \tag{66}$$

and the power spectras are obtained in the form

$$P_\zeta = \frac{2A^2cos^2(\eta+\vartheta)}{\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{2}}} cos^2\left(\sqrt{-\frac{2}{\eta^2}}\ \eta\ Log[\eta] + \vartheta\right), \tag{67}$$

$$P_h = \frac{2A^2}{\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{1}{2}}} cos^2\left(\sqrt{-\frac{2}{\eta^2}}\eta Log[\eta] + \vartheta\right). \tag{68}$$

The general expressions for the zeroth-order power spectra obtained in equations (56)-(57), (58)-(59), (64)-(65), and (67)-(68) for a radiation-dominated universe and a matter-dominated universe constitute the main results of this study.

The power spectrum is created by complicated but well understood Physics, depending not only on the perturbation spectrum, but also on the matter composition of the Universe. The observed oscillatory behaviour (compressions and rarefactions) is due to acoustic oscillations. “Acoustic” because the waves move with the sound speed. The plasma heats as it compresses and cools as it expands giving rise to the observed CMB temperature fluctuations [11].

Inflation is achieved when the universe is filled with a scalar field (the inflaton), whose slowly decreasing potential energy dominates the total energy of the universe. After inflation the universe, reheated while the inflaton decayed into particles and filled the earth with standard model particles, starting off the radiation dominated era of the universe.

Any form of matter satisfying the condition for inflation (accelerated expansion) given in equation (15), will cause an exponential growth/decay (acceleration) of the scale factor. This can be illustrated when the graphs for the power spectra are plotted.

The graphs of the evolution of the power spectrum with time obtained in this study both in a radiation-dominated and a matter-dominated universe, depicts a decaying oscillatory behaviour, which are the expected results. The varying heights of the peaks are due to the presence of an attractive form of gravity which causes more compression and less stretching. Therefore the odd peaks are higher (more compression) and the even peaks are lower (less stretching).

$$P_\zeta = \frac{1}{8\pi^2}\left|\frac{\mu_g}{Z_g}\right|^2 = \frac{A^2\cos^2(\eta+\theta)}{16\pi^2\beta^2\eta^2}$$

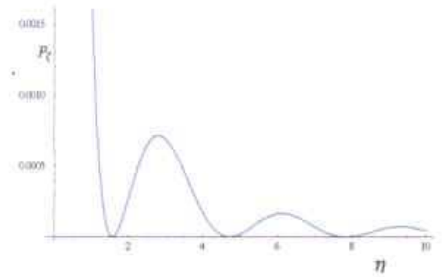

Figure 1a: Zeroth-order power spectra against conformal time η with k=$\pm$1, 0,over the range $\eta = 0 \rightarrow 10$ for a radiation-dominated universe.

$$P_\zeta = \frac{1}{8\pi^2}\left|\frac{\mu_g}{Z_g}\right|^2 = \frac{A^2\cos^2(\eta+\theta)}{16\pi^2\beta^2\eta^2}$$

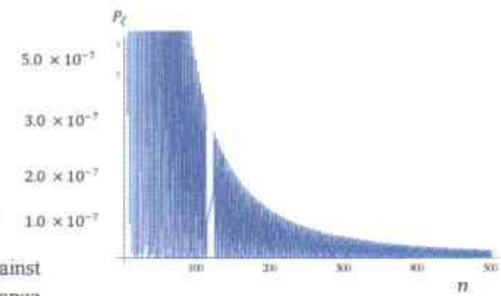

Figure 1b: Zeroth-order power spectra against conformal time η with k=$\pm$1, 0,over the range $\eta = 0 \rightarrow 500$ for a radiation-dominated universe.

$$P_\zeta = \frac{A^2}{12\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{3}{2}}}\cos^2\left(\frac{\sqrt{1-\frac{2}{\eta^2}}\,\eta\left(\sqrt{-2+\eta^2}+\sqrt{2}ArcTan\left[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}\right]\right)}{\sqrt{-2+\eta^2}}+\vartheta\right)$$

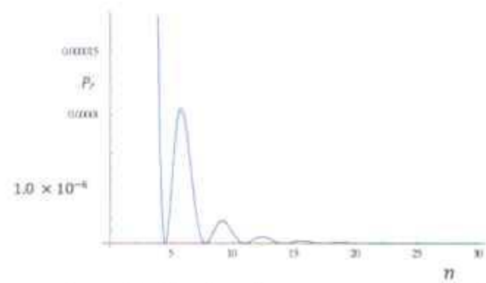

Figure 2a: Zeroth-order power spectra against conformal time η with k=$\pm$1, 0,over the range $\eta = 0 \rightarrow 30$ for a matter-dominated universe.

$$P_\zeta = \frac{A^2}{12\pi^2\beta^2\eta^4\left(1-\frac{2}{\eta^2}\right)^{\frac{3}{2}}}\cos^2\left(\frac{\sqrt{1-\frac{2}{\eta^2}}\,\eta\left(\sqrt{-2+\eta^2}+\sqrt{2}ArcTan\left[\frac{\sqrt{2}}{\sqrt{-2+\eta^2}}\right]\right)}{\sqrt{-2+\eta^2}}+\vartheta\right)$$

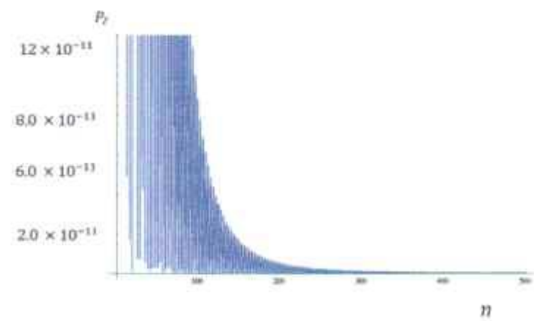

Figure 2b: Zeroth-order power spectra against conformal time η with k=$\pm$1, 0,over the range $\eta = 0 \rightarrow 500$ for a matter-dominated universe

Figures (1a) and (2a), depicts the decaying oscillatory behaviour. The rate of decay depends on the form of the scale factor for different eras of the universe, and is too rapid in a matter-dominated universe. The thick wave fronts in figures (1b) and (2b) are due to the high frequency range.

From the general expressions for the power spectra in equations (56) – (57), (58) – (59), (64) – (65), and (67) – (68), when the conformal time $\eta = 0$, the power spectra blows up. This corresponds to the singularity where all matter in the universe was concentrated at a point, at an initial time $t = 0$. This is called the Big Bang.

The exact shape of the universe is still a matter of debate in Physical Cosmology, but experimental data from various independent sources (Wilkinson Microwave Anistropy Probe (WMAP), Balloon Observations of Millimetric Extragalatictic Radiation ANd Geophysics (BOOMERanG), Planck for example) confirm that the universe is flat with only 0.4% of error [21] [22].

A flat universe $(k = 0)$ expands forever but at a continually decelerating rate, with expansion asymptotically approaching zero. With dark energy, the expansion rate of the universe initially slows down, due to the effect of gravity.

In a closed universe $(k = 1)$, gravity eventually stops the expansion of the universe, after which it starts to contract until all the matter in the universe collapses to a point, a final singularity termed the “Big Crunch”, the opposite of the Big Bang.

Even without dark energy, an open universe expands forever, with gravity negligibly slowing the rate of expansion. With dark energy, the expansion not only continues but accelerates [21].

The graphs of the power spectra against conformal time obtained in this study both in a radiation-dominated and a matter-dominated universe, are decreasing and are asymptotic to the horizontal axis, that is to say, the graphs do not get to zero.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1. Conclusion

We have provided a general approximate solution of arbitrary level of accuracy of the equation governing the evolution of cosmological perturbations in single field inflationary models through factorization and successive boost transformation of the equivalent matrix equation. The fact that the basic approximation parameter, starting with the zeroth-order parameter reduces progressively under successive boost operation from dynamical frames of higher accuracy has led to the concept of accuracy levels. Each dynamical frame represents an accuracy level, and an advancement from a frame of a lower accuracy to a frame of a higher accuracy level is achieved through a boost transformation or an appropriate succession of boost transformations.

The boost transformations effectively generate series expansions in terms of the derivative of the first-order derivatives of the approximation parameters. We have established that the factorization approximation parameter vanishes rapidly, and that the zeroth-order approximation or lowest accuracy dynamical frame is exactly the leading order/first-order WKB approximation.

The factorization and boost transformation procedure developed in this study is systematically extendable. The zeroth-order solution, is exactly the leading order/first- order standard WKB solution, and takes exactly the same form of the assumed solution (ansatz) in the standard WKB approximation. Furthermore, it does not require any matching condition about any particular point, that is to say, a turning point as outlined in the results for the WKB mode function and therefore the issue of divergence at a turning point as encountered in the WKB approximation does not arise in the approximation procedure developed in this study.

The general expressions for the zeroth-order power spectra obtained in equations (56)-(57), (58)-(59), (64)-(65), and (67)-(68) for a radiation-dominated universe and a matter-dominated universe constitute the main results of this study. The solution for the mode function obtained in this study, coincide exactly with the WKB approximation trial solutions.

5.2. Recommendations

The factorization and boost transformation procedure developed in this study is straightforward and quite effective in providing approximate solutions of the semi-classical model of the equation governing the evolution of cosmological perturbations in single field inflationary models.

The results obtained are very interesting, and therefore further comparison should be carried out with results of numerical calculations and experimental observations if specific parameters are substituted into the results as appropriate.

The approximation procedure is systematically extendable, once the time evolution operator Ʊ(η) is evaluated explicitly and substituted into equation (52e) to obtain $\bar{\phi}(\eta)$ and $\bar{\phi}^*(\eta)$, the desired general solution of the equation of dynamics governing the evolution of cosmological perturbations in single field inflationary scenario in equation (2) up to the $n^{th} - order$ accuracy follows easily using the definition of the mode function □(η) in equation (40a).

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