ON COMPLETELY POSITIVE MAPS

by

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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DEDICATION

To my dad and late mom. Through whose invaluable influence and guidance, I developed the right attitude and value for education.

ABSTRACT

Completely positive maps is an important field due to its significance, application and mathematics itself. While discussing the properties of the positive maps, researchers have questioned whether the properties of the positive maps also hold for completely positive maps. In chapter 1, we have started with a C^*-algebra $\mathcal{A}$, generated on $\mathcal{A}$ other C^*-algebras and then investigated these forms of C^*-algebras. We investigated whether the properties of $\mathcal{A}$ such as self-adjointedness and completeness under norm still hold on the C^*-algebras generated on $\mathcal{A}$. In chapter 2, the condition for the positivity of the elements of these generated C^*-algebras is given. This has been done by showing that their inner product with elements from a Hilbert space is positive. A unital contraction is necessarily positive. Conditions under which positive maps are completely positive are discussed. In chapter 3, boundedness and complete boundedness of these maps have been investigated. This, we have done by showing that, indeed, whenever the operator system is a C^*-algebra, then a positive map is bounded and completely bounded, if its norm is equal to its complete bound which must be finite. All completely positive maps are completely bounded, however the converse is not always true. This has been shown by giving examples and counter examples. The results of this study will pave way for construction of new C^*-algebras from the known ones, which will be helpful in the development of the research on positive maps on these generated C^*-algebras and may also be applied by mathematicians in solving spectral problems.

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Chapter 1

INTRODUCTION

In this chapter, literature review, definitions of some of the terms, theorems and propositions and in some cases examples that are essential in this study are given. Most of these have been obtained from the references, see [GK08, Mur90, Pau03, Tak79].

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{H}^{(n)}$ be the direct sum of $n-$copies of $\mathcal{H}$. If $M_n(B(\mathcal{H}))$, is the set of $n\times n$ matrices with entries from $B(\mathcal{H})$ and $B\left(\mathcal{H}^{(n)}\right)$ is the space of all bounded linear operators on $\mathcal{H}^{(n)}$, then it is shown that there exist linear maps $\phi : M_n(B(\mathcal{H})) \to B\left(\mathcal{H}^{(n)}\right)$ such that ϕ is a $*-$isomorphism, where $n \in \mathbb{N}$. Moreover, this ϕ is a representation of $M_n(B(\mathcal{H}))$ on the Hilbert space $\mathcal{H}^{(n)}$. Therefore, we can identify $M_n(B(\mathcal{H}))$ with $B\left(\mathcal{H}^{(n)}\right)$. Thus $M_n(B(\mathcal{H})) \cong B\left(\mathcal{H}^{(n)}\right)$. This identification gives us a unique norm that makes the $*$-algebra $M_n(B(\mathcal{H}))$ a C^*-algebra, see [Mur90].

If $\mathcal{A}$ is any C^*-algebra and $\phi : \mathcal{A} \to B(\mathcal{H})$ a $*$-homomorphism, then the **collection of norms** on $M_n(\phi(\mathcal{A}))$ is **independent** of the particular **representation** ϕ. By Gelfand Naimark Segal theorem, see [Mur90, Tak79], $\mathcal{A}$ is a closed self-adjoint subalgebra of $B(\mathcal{H})$.

1.1 Literature Review

The development of the theory of positive and completely positive maps are areas that have attracted a lot of interest from Mathematicians. In 1943, M.A. Naimark published two unrelated results. That is, the possibility of dilation of a positive operator valued measure to a spectral measure and the characterization of certain operator valued positive functions on groups in terms of representations on a larger space, see [Nai43a, Nai43b, Pau03]. A few years later, B. Sz.-Nagy obtained a theorem of unitary dilations of contractions on a Hilbert space, whose importance turned out to open a new and vast field of investigations of models of linear operators on Hilbert space in terms of a generalized Fourier analysis, see [NF70, Mur90, Pau03]. One of the better known dilation theorems is due to Sz.-Nagy and asserts that every contraction operator can be dilated to a unitary operator. The most famous application of this idea is Sz.-Nagy's elegant proof of an inequality of von Neumann to the effect that the norm of a polynomial in a contraction operator is at most the supremum of the absolute value of the polynomial over the unit disk, in this way revealing its spectral character, see [NF70, Pau03]. In 1955, W.F. Stinespring obtained a theorem characterizing certain operator valued positive maps on C^*-algebras in terms of representations of those C^*-algebras, what is called Stinespring Representation, see [NF70, Pau03, Sti55]. M.A. Naimark showed that every C^*-algebra can be faithfully represented as a subalgebra of

$B(\mathcal{H})$. Every representation π of $\mathcal{A}$ on $B(\mathcal{H})$ and vector $x \in \mathcal{H}$ defines a linear functional f on $\mathcal{A}$ by $f(a) = \langle \pi(a)x, x \rangle$. Such a functional is positive, (and is automatically continuous and contractive). There exists a Hilbert space $\mathcal{H}_f$, a vector $x_f \in \mathcal{H}_f$ and a representation π_f of $\mathcal{A}$ on $\mathcal{H}_f$ such that $f(a) = \langle \pi_f(a)x_f, x_f \rangle \quad \forall \quad a \in \mathcal{A}$. This construction is known as the Gelfand-Naimark-Segal (GNS) construction, see [Mur90, Nai43a, Nai43b, Pau03]. W.F. Stinespring introduced the theory of positive maps as a means of giving abstract necessary and sufficient conditions for the existence of dilations, a technique for studying operators on a Hilbert space ($\mathcal{H}$) by representing a given operator, say T_1 as the restriction of a (hopefully) better understood operator, say T_2, acting on a larger Hilbert space, to the original space. He showed that completely positive maps always have a representation of the form $\pi_2[\phi(\mathcal{A})] = V^*\pi_1(\mathcal{A})V$, where π_1 and π_2 are representations of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{A}$ respectively, that is, $\pi_1 : \mathcal{A}_1 \to B(\mathcal{H})$ and $\pi_2 : \mathcal{A}_2 \to B(\mathcal{H})$, ϕ is a completely positive operator and V is a bounded operator from $\mathcal{H}$ to another Hilbert space say, $\mathcal{K} \subseteq \mathcal{A}$, see [Pau03, Sti55].

Theorem 1.1.1 (The Stinespring Representation Theorem)
Let $\mathcal{A}$ be a unital C^*-algebra and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be a completely positive map, then there exists a Hilbert space $\mathcal{K}$, a bounded operator $V : \mathcal{H} \to \mathcal{K}$ and a unital $*$-homomorphism, $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\phi(a) = V^*\pi(a)V$, for every $a \in \mathcal{A}$.

In this theorem, we also have that, $\|\phi\|_{cb} = \|\phi(1)\| = \|V^*V\| = \|V\|^2$, see [Pau03]. This theorem opened a large field of investigations on a new concept in operator algebra that is now called complete positivity, mainly due to the pioneering work of M.D. Choi, see [Cho75, Cho72, Pau03, Sti55]. The connections between completely positive maps and dilation theory were broadened further by Arverson , who developed a deep structure theory for these maps. He showed that, if $\mathcal{S} \subseteq \mathcal{A}$ is an operator system and $\phi : \mathcal{S} \longrightarrow \mathcal{B}(\mathcal{H})$ is completely positive, then there exists a completely positive map $\psi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ that extends ϕ such that $\psi(a) = \phi(a)$ for every $a \in \mathcal{S}$, see [Pau03, Arv76, Arv69].

This result by Arverson yielded another result due to Wittstock, who worked on operator spaces instead of operator systems.

Theorem 1.1.2

*(**Wittstock's Extension Theorem**) Let $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and $\phi : \mathcal{M} \longrightarrow \mathcal{B}(\mathcal{H})$ be completely bounded, then there exists a completely bounded map $\psi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ that extends ϕ and satisfies $\|\phi\|_{cb} = \|\psi\|_{cb}$, see [Pau03].*

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map, then

$$\|\phi\| = \sup\{\|\phi(a)\| : \|a\| \leq 1\}, \ \forall\, a \in \mathcal{A}.$$

We can define the maps $\phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ for all $[a_{i,j}] \in M_n(\mathcal{A})$. Then,

$$\|\phi_n\| = \sup\{\|\phi_n([a_{i,j}])\| : n \in \mathbb{N}; \|a_{i,j}\| \leq 1\}$$

and

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\}.$$

Wittstock and Haagerup generalised Stinespring Representation Theorem to completely bounded maps. The generalised Stinespring theorem states that if $\mathcal{A}$ is a unital C^*-algebra and $\phi : \mathcal{A} \to B(\mathcal{H})$ is a completely bounded map, then there exists a Hilbert space $\mathcal{H}'$, operators $V, W \in B(\mathcal{H}; \mathcal{H}')$ and a unital $*$-homomorphism (a $*$-representation), $\pi : \mathcal{A} \to B(\mathcal{H}')$, such that $\|V\|\|W\| = \|\phi\|_{cb}$ and $\phi(a) = W^*\pi(a)V,\ a \in \mathcal{A}$.

In the early 1980's, researchers began extending much of the theory of completely positive maps to the family of completely bounded maps, completely positive maps and completely bounded maps as the analogue of positive measures and bounded measures respectively, see [Pau03]. To discuss completely positive maps and completely bounded maps between two spaces, their domains and ranges need to be operator system and operator space, respectively. Such spaces arise naturally as subspaces of the space of bounded operators on a Hilbert space. However results of Choi, Effros and Ruan gave abstract characterizations of operator systems and operator spaces that enabled researchers to treat their theory and the corresponding theories of completely positive maps and completely bounded maps in a way that was free of dependence on this underlying Hilbert space. These characterizations have had an impact on this field similar to the impact of the Gelfand-Naimark-Segal theorem on the C^*-algebras and led to a deeper understanding of many results of C^*-algebras and Von Neumann algebras, see [ER91, Mur90, Cho75, Cho72, Pau03]. The aim of this work was to study the properties of completely positive maps. In particular, the conditions under which positivity imply complete positivity and when a map is completely bounded.

1.2 Statement Of The Problem

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map, we define maps $\phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$, by the formula

$$\phi_n([a_{ij}]) = [\phi(a_{ij})], \ \forall \, n \in \mathbb{N}, \ i, j = 1, ..., n,$$

where $[a_{i,j}] \in M_n(\mathcal{A})$ and $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ are C^*-algebras of $n \times n$ matrices with entries from $\mathcal{A}$ and $\mathcal{B}$. Properties of ϕ_n have not been exhaustively studied. We have investigated some properties of positive maps, of interest is on how assumptions of positivity of ϕ is related to its norm and conversely, when the norm of ϕ guarantees that it is positive. It is not true that every positive map is completely positive or every bounded map is completely bounded. In this study, we investigated some of the elementary properties of these classes of maps and determined conditions when positive maps are automatically completely positive. Further, we have investigated conditions for which ϕ is completely bounded.

1.3 Objectives Of The Study

The purpose of this study was to do the following:

1. Establish how the notion of positivity is introduced to the C^*-algebras

2. Investigate some of the elementary properties of completely positive maps and determine conditions when positive maps are automati-

cally completely positive.

3. Investigate conditions for which a map is completely bounded.

1.4 Significance Of The Study

The results of this study would pave way for construction of new C^*-algebras from the known ones, which would be helpful in the development of the research on positive maps on these generated C^*-algebras and may also be applied by mathematicians in solving spectral problems.

1.5 Research Methodology

In this study, we have determined when positive maps are automatically completely positive. That is, given that ϕ is positive, we have determined the properties for which ϕ_n is positive for every n. Further, the conditions for which a map is completely bounded have been investigated. By showing that the estimation $\|\phi_n\| \leq n\|\phi\| < \infty$ is sharp for all $n \in \mathbb{N}$, has proved the complete boundedness of ϕ. To show that a completely positive map is completely bounded, this study has proved the equality $\|\phi\| = \|\phi(I)\| = \|\phi_n\| = \|\phi\|_{cb} < \infty$ and that $\phi^* = \phi$.

1.6 C^*-algebras

Most of the literature and results in this thesis have been obtained from the references see [Mur90, Cho75, Pau03, Tak79], from which the proofs of the results may also be obtained.

1.6.1 Algebra

Definition 1.6.1

An **algebra** over a field $\mathbb{K}$ is a vector space $\mathcal{A}$ together with a bilinear map (vector multiplication) $\mathcal{A}^2 \to \mathcal{A},\ (a, b) \mapsto ab$, such that

1) $a(bc) = (ab)c \qquad (a, b, c \in \mathcal{A});$

2) $(\alpha a + \beta b)c = \alpha ac + \beta bc$ and $c(\alpha a + \beta b) = \alpha ca + \beta cb\ \ (\alpha, \beta \in \mathbb{C})$.

Definition 1.6.2

A **subalgebra** of $\mathcal{A}$ is a vector subspace $\mathcal{B}$ such that

$$b, b' \in \mathcal{B} \quad \Rightarrow \quad bb' \in \mathcal{B}.$$

Definition 1.6.3

A **normed algebra** $\mathcal{A}$ is a vector space with a norm defined on it.

A norm $\|.\|$ on $\mathcal{A}$ is said to be multiplicative if

$$\|ab\| \leq \|a\|\|b\| \quad a, b \in \mathcal{A}.$$

The pair $(\mathcal{A}, \|.\|)$ is called a **normed algebra**. If $\mathcal{A}$ admits a unit e $(ae = ea = a,\ \forall\ a \in \mathcal{A})$ and $\|e\| = e$, then $\mathcal{A}$ is a **unital normed** algebra.

Definition 1.6.4

A **Banach algebra** is a complete normed algebra. A complete unital normed algebra is called a **unital Banach algebra.**

Example 1.6.5

Let X be a Banach space, then set $B(X)$ of all bounded linear maps (operators) from X to itself is a normed algebra where the operations are defined pointwise for addition and scalar multiplication, multiplication given by $(u, v) \mapsto u \circ v$, and

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|u(x)\|.$$

Since X is a Banach space, $B(X)$ is complete and therefore it is a Banach algebra.

Example 1.6.6

The algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with entries in $\mathbb{C}$ is identified with $B(\mathbb{C}^n)$. It is therefore a unital Banach algebra. Upper and lower triangular matrices are subalgebra of $M_n(\mathbb{C})$.

Definition 1.6.7

An element a of a Banach algebra $\mathcal{A}$ is invertible if there is an element $b \in \mathcal{A}$ such that $ab = ba = 1, \quad b = a^{-1}$ is unique.

$$Inv(\mathcal{A}) = \{a \in \mathcal{A} \mid a \textit{ is invertible}\}.$$

Definition 1.6.8

The **spectrum** of an element a is the set

$$\begin{aligned} \sigma(a) = \sigma_{\mathcal{A}}(a) &= \{\lambda \in \mathbb{C} | \lambda I - a \notin Inv(\mathcal{A})\}. \\ &= \{\lambda \in \mathbb{C} | a - \lambda I \in Inv(\mathcal{A})\}. \end{aligned}$$

Theorem 1.6.9

(Spectrum). Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$. Then $\sigma(a)$ is a

nonempty compact subset of $\{z : |z| \leq \|a\|\}$.

Definition 1.6.10

The spectral radius $r(a)$ of an element $a \in \mathcal{A}$ is given by

$$r(a) = sup\{|\lambda| : \lambda \in \sigma(a)\}$$

Definition 1.6.11

An **involution** on an algebra $\mathcal{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathcal{A}$, such that

(i) $a^{**} = a$

(ii) $(ab)^* = b^* a^*$ and

(iii) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^* \quad \forall\, a, b \in \mathcal{A}$.

The pair $(\mathcal{A}, *)$ is called an **involutive algebra**, or a $*$-algebra.

Definition 1.6.12

An element $a \in \mathcal{A}$ is **self-adjoint** or **hermitian** if $a = a^*$. For each $a \in \mathcal{A}$, there exists unique hermitian elements $b, c \in \mathcal{A}$ such that $a = b + ic$ where $b = \frac{1}{2}(a + a^*)$ and $c = \frac{1}{2i}(a - a^*)$. The elements a^*a and aa^* are hermitian. Denote the set of hermitian elements of $\mathcal{A}$ by $\mathcal{A}_{sa}$. a is **normal** if $a^*a = aa^*$.

If $\mathcal{A}$ is unital, then $e^* = e$ $(e^* = (ee^*)^* = e)$. If $a \in Inv(\mathcal{A})$, then $(a^*)^{-1} = (a^{-1})^*$. Hence

$$\sigma(a^*) = \sigma(a)^* = \{\bar{\lambda} \in \mathbb{C} | \lambda \in \sigma(a)\}.$$

An element $a \in \mathcal{A}$ is a **unitary** if $a^*a = aa^* = e$. If $a^*a = e$, then a is an isometry, and if $aa^* = e$ then a is a co-isometry.

Definition 1.6.13

A **Banach $*$-algebra** is a $*$ -algebra $\mathcal{A}$ together with a complete sub-multiplicative norm such that

$$\|a^*\| = \|a\| \ \ (a \in \mathcal{A}).$$

Definition 1.6.14

A **C$^*-$algebra** is a Banach $*$-algebra such that

$$\|a^*a\| = \|a\|^2 \quad (a \in \mathcal{A}).$$

A closed $*$-subalgebra of a C^*-algebra is a **C$^*-$subalgebra**.

Example 1.6.15

The scalar field $\mathbb{C}$ is a unital C^*-algebra with involution given by complex conjugation $\lambda \mapsto \bar{\lambda}$.

Example 1.6.16

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is C^*-algebra. (Gelfand Naimark Segul theorem).

Theorem 1.6.17

If a is a self-adjoint element of a C^*-algebra $\mathcal{A}$, then $r(a) = \|a\|$, [3.3]

Corollary 1.6.18

There is at most one norm on a $*-$algebra making it a C^*-algebra.

Lemma 1.6.19

Let $\mathcal{A}$ be a Banach algebra endowed with involution such that $\|a\|^2 \leq \|a^*a\|$ $(a \in \mathcal{A})$. Then $\mathcal{A}$ is a C^*-algebra.

1.6.2 Positive Elements Of C^*-Algebra.

Definition 1.6.20

An element a of a C^*-algebra $\mathcal{A}$ is positive, if and only if a is of the form $a = b^*b$ for some $b \in \mathcal{A}$ or if a is hermitian and $\sigma(a) \subseteq \mathbb{R}^+$. Denote by $\mathcal{A}^+$ the set of positive elements of $\mathcal{A}$. In this thesis, the symbol $a \geq 0$ will be used to denote an element a which is positive.

Definition 1.6.21

A $*$**-homomorphism** from $*$-algebra $\mathcal{A}$ to $*$-algebra $\mathcal{B}$ is a linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that

(i) $\phi(ab) = \phi(a)\phi(b)$

(ii) $\phi(a^*) = \phi(a)^*, \ \forall \ a, b \in \mathcal{A}.$

ϕ is unital if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi(I_{\mathcal{A}}) = I_{\mathcal{B}},$ that is $\phi(1) = 1.$

If in addition, ϕ is a bijection, it is a $*$**-isomorphism**.

Theorem 1.6.22

A $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ from a Banach $*$-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ is necessarily norm decreasing.

PROOF. We suppose that $\mathcal{A}, \mathcal{B}$ and φ are unital. If $a \in \mathcal{A}$, then $\sigma(\varphi(a)) \ \subseteq \ \sigma(a)$, so

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| \leq \|a\|^2.$$

Taking the square root of both sides, we have

$$\|\varphi(a)\| \leq \|a\|. \tag{1.1}$$

□ From inequality 1.1, we get that φ is a contraction.

$$\begin{aligned}\|\varphi\| &= \sup_{\|a\|\leq 1} \|\varphi(a)\| \\ &\leq \sup_{\|a\|\leq 1} \|a\| = 1\end{aligned}$$

Thus, $\|\varphi\| \leq 1$.

Theorem 1.6.23

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.

1. *If $u \in B(\mathcal{H}_1, \mathcal{H}_2)$, then there is a unique element $u^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle u(x_1), x_2\rangle = \langle x_1, u^*(x_2)\rangle, \quad x_1 \in \mathcal{H}_1, \quad x_2 \in \mathcal{H}_2.$$

2. *The map $u \mapsto u^*$ is conjugate-linear and $u^{**} = u$. Also*

$$\|u\| = \|u^*\| = \|u^*u\|^{1/2}.$$

1.6.3 Positive Linear Functionals

If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map between C^*-algebras, it is said to be positive if ϕ maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. In this case $\phi(\mathcal{A}_{sa}) \subseteq \mathcal{B}_{sa}$.

Theorem 1.6.24

If τ is a positive linear functional on a C^*-algebra $\mathcal{A}$, then it is bounded.

Proof. If τ is not bounded, then $\sup_{a\in\mathcal{S}} \tau(a) = +\infty$; $\mathcal{S}$ is the set of all positive elements of $\mathcal{A}$ of norm not greater than 1. Hence there is a sequence (a_n) in $\mathcal{S}$ such that $2^n \leq \tau(a_n)$, $\forall\, n \in \mathbb{N}$. Set

$$a = \sum_{n=0}^{\infty} a_n/2^n \Rightarrow a \in \mathcal{A}^+.$$

$1 \leq \tau(a_n/2^n)$, therefore

$$N \leq \sum_{n=0}^{N-1} \tau\left(a_n/2^n\right) = \tau\left(\sum_{n=0}^{N-1} a_n/2^n\right) \leq \tau(a).$$

Hence, $\tau(a)$ is an upper bound for the set $\mathbb{N}$ which is impossible. Thus τ is bounded. □

Theorem 1.6.25

Let τ be a bounded linear functional on a C^*-algebra $\mathcal{A}$. The following conditions are equivalent:

1. *τ is positive .*
2. *For each approximate unit $(u_\lambda)_{\lambda\in\bigwedge}$ of $\mathcal{A}$, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*
3. *For some approximate unit $(u_\lambda)_{\lambda\in\bigwedge}$ of $\mathcal{A}$, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*

Corollary 1.6.26

If τ is a bounded linear functional on a unital C^*-algebra, then τ is positive iff $\tau(1) = \|\tau\|$.

Definition 1.6.27

A **state** on a C^*-algebra $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$ of norm 1. Denote by $\mathcal{S}(\mathcal{A})$ the set of all states of $\mathcal{A}$.

Theorem 1.6.28

If a is a normal element of a non-zero C^*-algebra $\mathcal{A}$, then there is a state τ of $\mathcal{A}$ such that $\|a\| = |\tau(a)|$.

Theorem 1.6.29

Let a be a self-adjoint element of a C^*-algebra $\mathcal{A}$. Then $a \in \mathcal{A}^+$ if and only if $\tau(a) \geq 0$, $\forall$ positive linear functionals τ on $\mathcal{A}$.

Proof. The forward implication is plain. Suppose conversely that $\tau(a) \geq 0$ for all positive linear functionals τ on $\mathcal{A}$. Let $(\mathcal{H}, \phi)$ be the universal representation of $\mathcal{A}$, and let $x \in \mathcal{H}$. Then the linear functional

$$\tau : \mathcal{A} \to \mathbb{C},\ b \mapsto \langle \phi(b)(x), x \rangle \,,$$

is positive, so $\tau(a) \geq 0$; that is, $\langle \phi(a)(x), x \rangle \geq 0$. Since this is true $\forall \ x \in \mathcal{H}$, and since $\phi(a)$ is self-adjoint, therefore $\phi(a)$ is a positive operator on $\mathcal{H}$. Hence, $\phi(a) \in \phi(\mathcal{A})^+$, so $a \in \mathcal{A}^+$, because the map $\phi : \mathcal{A} \to \phi_{\mathcal{A}}$ is a $*$-isomorphism. □

1.6.4 The Gelfand-Naimark Representation

Definition 1.6.30

A **representation** of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \varphi)$, where $\mathcal{H}$ is a Hilbert space and $\varphi : \mathcal{A} \to B(\mathcal{H})$ is a $*$-homomorphism. $(\mathcal{H}, \varphi)$ is **faithful** if φ is injective.

With each positive linear functional there is associated a representation. Suppose that τ is a positive linear functional on a C^*-algebra $\mathcal{A}$. Setting

$$\mathcal{N}_\tau = \{a \in \mathcal{A} \mid \tau(a^*a) = 0\} \,,$$

$\mathcal{N}_\tau$ is a closed left ideal of $\mathcal{A}$ and the map

$$(\mathcal{A}/\mathcal{N}_\tau)^2 \to \mathbb{C}, (a+\mathcal{N}_\tau, b+\mathcal{N}_\tau) \mapsto \tau(b^*a),$$

is a well-defined inner product on $\mathcal{A}/\mathcal{N}_\tau$. Denote by $\mathcal{H}_\tau$ the Hilbert completion of $\mathcal{A}/\mathcal{N}_\tau$. If $a \in \mathcal{A}$, define an operator $\varphi(a) \in B(\mathcal{A}/\mathcal{N}_\tau)$ by setting

$$\varphi(a)(b+\mathcal{N}_\tau) = ab+\mathcal{N}_\tau.$$

$\varphi(a)$ is clearly linear and bounded, i.e. $\|\varphi(a)\| \leq \|a\|$ holds. Since

$$\|\varphi(a)(b+\mathcal{N}_\tau)\|^2 = \tau(b^*a^*ab) \leq \|a\|^2\tau(b^*b) = \|a\|^2\|b+\mathcal{N}_\tau\|^2.$$

The operator $\varphi(a)$ has a unique extension to a bounded operator $\varphi_\tau(a)$ on $\mathcal{H}_\tau$. The map

$$\varphi_\tau : \mathcal{A} \to B(\mathcal{H}_\tau),\ a \mapsto \varphi_\tau(a)$$

is a $*$-homomorphism. The representation $(\mathcal{H}_\tau, \varphi_\tau)$ of $\mathcal{A}$ is the **Gelfand-Naimark-Segal** representation (GNS) associated to τ. If $\mathcal{A}$ is non-zero, its universal representation is the direct sum of all the representations $(\mathcal{H}_\tau, \varphi_\tau)$, where τ range over $\mathcal{S}$.

This shows that every C^*-algebra can be regarded as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ as indicated in the theorem 1.6.31 below. See [Mur90, Pau03]

Theorem 1.6.31

(Gelfand-Naimark). If $\mathcal{A}$ is a C^*-algebra, then it has a faithful representation. Specifically, its ***universal representation*** *is faithful.*

Most of the results above were obtained from [Mur90, Tak79]

1.7 Matrices of C*-algebra

Most of the results in this section were obtained from [Pau07, Pau03].

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$ and $M_n(B(\mathcal{H}))$, the set of all $n \times n$ matrices with entries from $B(\mathcal{H})$. We show that $M_n(B(\mathcal{H}))$ is a $*$-algebra.

We first show that $M_n(B(\mathcal{H}))$ is an algebra, that is, it is associative and linear. Let $a = (a_{i,j}),\ b = (b_{i,j}),\ c = (c_{i,j}) \in M_n(B(\mathcal{H})),\ i\ ,\ j\ =\ 1\ ,\ ...\ ,\ n.$ Define

$$(ab)_{i,j} = (a_{i,j}).(b_{i,j}) = \sum_{k=1}^{n} a_{i,k} b_{k,j},$$

then we have

$$\begin{aligned}
((ab)c)_{i,j} = \sum_{k=1}^{n} (ab)_{i,k}\, c_{k,j} &= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{i,l} b_{l,k} c_{k,j} \\
&= \sum_{l=1}^{n} a_{i,l} \sum_{k=1}^{n} b_{l,k} c_{k,j} \\
&= \sum_{l=1}^{n} a_{i,l} (bc)_{l,j} = (a(bc))_{i,j}
\end{aligned}$$

Thus, $M_n(B(\mathcal{H}))$ is associative. Linearity follows from multiplication of a matrix by scalar. Hence $M_n(B(\mathcal{H}))$ is an algebra.

Infact, $M_n(B(\mathcal{H}))$ is a $*$-algebra, if involution is defined as

$$(a_{i,j})^* = a_{j,i}^*.$$

Now consider $\mathcal{H}^{(n)}$ the orthogonal (or direct) sum of n copies of $\mathcal{H}$ with an orthogonal basis, then there is a norm and inner product on $\mathcal{H}^{(n)}$ that makes it into a Hilbert space. That is,

$$\langle h,k\rangle = \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} = \langle h_1,k_1\rangle_{\mathcal{H}} + ... + \langle h_n,k_n\rangle_{\mathcal{H}},$$

where

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \mathcal{H}^{(n)}.$$

$$\|h\|^2 = \left\| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \|h_1\|^2 + ... + \|h_n\|^2$$

We prove that an element of $M_n(B(\mathcal{H}))$ is a linear map on $\mathcal{H}^{(n)}$. Indeed, for

$T = [T_{ij}] \in M_n(B(\mathcal{H})), h \in \mathcal{H}^{(n)}$, we have the definition

$$[T_{ij}]h = \begin{pmatrix} T_{11} & T_{12} & . & . & . & T_{1n} \\ T_{21} & T_{22} & . & . & . & T_{2n} \\ \vdots & & & & & \\ T_{n1} & T_{n2} & . & . & . & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n T_{1j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix}.$$

Let $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \mathcal{H}^{(n)}$ and α, β be any scalars, then

$$\begin{aligned}
[T_{ij}]\left(\alpha \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + \beta \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}\right) &= [T_{ij}]\left(\begin{pmatrix} \alpha h_1 \\ \vdots \\ \alpha h_n \end{pmatrix} + \begin{pmatrix} \beta k_1 \\ \vdots \\ \beta k_n \end{pmatrix}\right) \\
&= [T_{ij}]\begin{pmatrix} \alpha h_1 + \beta k_1 \\ \vdots \\ \alpha h_n + \beta k_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n T_{1j}\alpha h_j + \sum_{j=1}^n T_{1j}\beta k_j \\ \vdots \\ \sum_{j=1}^n T_{nj}\alpha h_j + \sum_{j=1}^n T_{nj}\beta h_j \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n T_{1j}\alpha h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}\alpha h_j \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n T_{1j}\beta k_j \\ \vdots \\ \sum_{j=1}^n T_{nj}\beta h_j \end{pmatrix} \\
&= \alpha \begin{pmatrix} \sum_{j=1}^n T_{1j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix} + \beta \begin{pmatrix} \sum_{j=1}^n T_{1j}k_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix}
\end{aligned}$$

$$= \alpha \left([T_{ij}] \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right) + \beta \left([T_{ij}] \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right)$$

Thus $[T_{ij}]$ is linear.

1.7.1 Operator norm

Definition 1.7.1

Let $T : X \to X$ be an operator from the vector space X into itself, then the norm of T is given by

$$\|T\| = sup\{\|Tx\| : \|x\| \leq 1, x \in X\}$$

We prove that the norm on $M_n(B(\mathcal{H}))$ can be approximated as

$$\|[T_{ij}]\| \leq \sqrt{\sum_{i,j=1}^{n} \|T_{i,j}\|^2}.$$

We show that $[T_{ij}]$ is bounded, i.e., there exists a $c \in \mathbb{R}^+$ such that $\|[T_{ij}]h\| \leq c\|h\|$. Letting $\|h\| \leq 1$, we have

$$\|[T_{ij}]h\|^2 = \left\langle \begin{pmatrix} T_{11} & . & . & . & T_{1n} \\ \vdots & & & & \\ T_{n1} & . & . & . & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_1 \end{pmatrix}, \begin{pmatrix} T_{11} & . & . & . & T_{1n} \\ \vdots & & & & \\ T_{n1} & . & . & . & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_1 \end{pmatrix} \right\rangle$$

$$
\begin{aligned}
&= \left\langle \begin{pmatrix} \sum_{j=1}^{n} T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^{n} T_{n,j}h_j \end{pmatrix}, \begin{pmatrix} \sum_{j=1}^{n} T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^{n} T_{n,j}h_j \end{pmatrix} \right\rangle \\
&= \left\| \sum_{j=1}^{n} T_{1,j}h_j \right\|^2 + \ldots + \left\| \sum_{j=1}^{n} T_{n,j}h_j \right\|^2 \\
&= \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} T_{i,j}h_j \right\|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|T_{i,j}h_j\|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|T_{i,j}\|^2 \|h_j\|^2 \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|T_{i,j}\|^2 \right) \sum_{j=1}^{n} \|h_j\|^2 \\
&= \left(\sum_{i,j=1}^{n} \|T_{i,j}\|^2 \right) \sum_{j=1}^{n} \|h_j\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\langle \sum_{j=1}^{n} T_{1,j}h_j, \sum_{j=1}^{n} T_{1,j}h_j \right\rangle + \ldots + \left\langle \sum_{j=1}^{n} T_{n,j}h_j, \sum_{j=1}^{n} T_{n,j}h_j \right\rangle \\
&= \left\| \sum_{j=1}^{n} T_{1,j}h_j \right\|^2 + \ldots + \left\| \sum_{j=1}^{n} T_{n,j}h_j \right\|^2 \\
&= \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} T_{i,j}h_j \right\|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|T_{i,j}h_j\|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|T_{i,j}\|^2 \|h_j\|^2
\end{aligned}
$$

$$\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\|T_{i,j}\|^2\right)\sum_{j=1}^{n}\|h_j\|^2$$
$$= \left(\sum_{i,j=1}^{n}\|T_{i,j}\|^2\right)\sum_{j=1}^{n}\|h_j\|^2$$

Taking the squareroot of both sides we have,

$$\|[T_{ij}]h\| \leq \sqrt{\sum_{i,j=1}^{n}\|T_{i,j}\|^2\sum_{j=1}^{n}\|h_j\|^2} = \sqrt{\sum_{i,j=1}^{n}\|T_{i,j}\|^2}\sqrt{\sum_{j=1}^{n}\|h_j\|^2} = c\|h\|,$$

where $c = \sqrt{\sum_{i,j=1}^{n}\|T_{i,j}\|^2}$.
Since $\sum_{j=1}^{n}\|h_j\|^2 = 1$, we have $\|[T_{ij}]\| \leq \sqrt{\sum_{i,j=1}^{n}\|T_{i,j}\|^2}$. Since $[T_{ij}]$ was arbitrarily chosen, we have

$$\|[T_{ij}]\| \leq \sqrt{\sum_{i,j=1}^{n}\|T_{i,j}\|^2},$$

Lemma 1.7.2

Let $\mathcal{H}$ be a Hilbert space and $\phi : M_n(B(\mathcal{H})) \to B(\mathcal{H}^{(n)})$ be a linear map from the C^*-algebra $M_n(B(\mathcal{H}))$ to the C^*-algebra $B(\mathcal{H}^{(n)})$, then

$$\max_{i,j}\|T_{ij}\| \leq \|[T_{ij}]\| \leq \left(\sum_{i,j=1}^{n}\|T_{ij}\|^2\right)^{\frac{1}{2}} \leq n\max_{i,j}\|T_{ij}\|, \;\; \forall\;\; T_{ij} \in M_n(B(\mathcal{H}))$$

Proof. Define a projection $E_i : \mathcal{H} \to \mathcal{H}^{(n)}$ by $E_i x = 0 \oplus 0 \oplus ...x... \oplus 0 \oplus ... \oplus 0$, where $x \in \mathcal{H}$. Therefore, $E_i^* E_i = I_{\mathcal{H}}$, $E_j^* E_i = 0$ and $\sum_{i=1}^{n} E_i^* E_i = I_{\mathcal{H}^n},\;\; i = 1, ..., n$. Clearly, $\|E_i\| = 1$.

Now,

$$T_{ij} = E_i^*[T_{ij}]E_j.$$

Therefore,

$$\|T_{ij}\| = \|E_i^*[T_{ij}]E_j\| \leq \|E_i^*\|\|[T_{ij}]\|\|E_j\| \leq \|[T_{ij}]\|.$$

This implies that

$$\max_{i,j} \|T_{ij}\| \leq \|[T_{ij}]\|. \tag{1.2}$$

Let $h \in \mathbb{H}$, then

$$\begin{aligned}
\|[T_{ij}]h\|^2 &= \left| \left\langle \begin{pmatrix} T_{11} & . & . & . & T_{1n} \\ \vdots & & & & \\ T_{n1} & . & . & . & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_1 \end{pmatrix}, \begin{pmatrix} T_{11} & . & . & . & T_{1n} \\ \vdots & & & & \\ T_{n1} & . & . & . & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_1 \end{pmatrix} \right\rangle \right|^2 \\
&= \left| \left\langle \begin{pmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{pmatrix}, \begin{pmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{pmatrix} \right\rangle \right|^2 \\
&\leq \sum_{j=1}^n \left\langle \sum_{j=1}^n |T_{k,j}h_j, T_{k,j}h_j|^2 \right\rangle \\
&\leq \sum_{j=1}^n \sum_{j=1}^n \|T_{k,j}\|\|h_j\|\|T_{k,j}\|\|h_j\| \\
&= \sum_{i=1}^n \sum_{j=1}^n \|T_{k,j}\|^2\|h_j\|^2 \\
&= \sum_{i=1}^n \sum_{j=1}^n (\|T_{k,j}\|\|h_j\|)^2 \\
&\leq \sum_{i=1}^n \left(\sum_{j=1}^n \|T_{k,j}\|^2 \right)^{1/2} \left(\sum_{j=1}^n \|h_j\|^2 \right)^{1/2} \textit{ by Schwarz inequality.}
\end{aligned}$$

$$= \sum_{i,j=1}^{n} \|T_{i,j}\|^2 \sum_{j=1}^{n} \|h_j\|^2$$

Taking the supremum with $\|h_j\| \leq 1$, we get $\|[T_{ij}]\|^2 \leq \sum_{i,j=1}^{n} \|T_{i,j}\|^2$. Taking the squareroot of both sides, we have

$$\|[T_{ij}]\| \leq \left(\sum_{i,j=1}^{n} \|T_{ij}\|^2 \right)^{\frac{1}{2}}. \tag{1.3}$$

Since $\|[T_{ij}]\| = \sup\{\|[T_{ij}]h\| : \|h\| \leq 1\}$, We also have that,

$$\left(\sum_{i,j=1}^{n} \|T_{ij}\|^2 \right)^{\frac{1}{2}} \leq n \max_{i,j} \|T_{ij}\|. \tag{1.4}$$

From the equations (1.2), (3.1) and (3.2) the assertion follows.

$\square$

Proposition 1.7.3

Let $M_n(B(\mathcal{H}))$ and $B\left(\mathcal{H}^{(n)}\right)$ be C^*-algebras, then there exists a linear map $\phi : M_n(B(\mathcal{H})) \to B\left(\mathcal{H}^{(n)}\right)$ such ϕ is a $*-$isomorphism.

To prove this proposition, we need to show the following :

- ϕ is linear
- ϕ is bijective, i.e both injective and surjective
- ϕ is homomorphic and that
- $\phi([T_{ij}]^*) = [T_{ij}]^*, \;\; \forall \;\; [T_{ij}] \in M_n(B(\mathcal{H}))$, i.e it preserves the adjoint.

Let $[T_{ij}] \in M_n(B(\mathcal{H}))$, we define the map by $\phi([T_{ij}]) = [T_{ij}]$. Set

$$
\begin{aligned}
\phi([T_{ij}])\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} &= \phi\begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{pmatrix} \\
&= \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \forall \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^{(n)} \\
. &= \begin{pmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{pmatrix} \\
&= \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \forall \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^{(n)} \\
.
\end{aligned}
$$

Thus ϕ is just the ordinary matrix multiplication.

We show the **linearity** of ϕ. Let $\alpha, \beta \in \mathbb{C}, [T_{i,j}], [S_{i,j}] \in M_n(B(\mathcal{H}))$ and $h \in \mathcal{H}^{(n)}$, then

$$
\phi(\alpha[T_{i,j}] + \beta[S_{i,j}])(h) = \begin{pmatrix} \Sigma_{j=1}^n(\alpha T_{1,j} + \beta S_{1,j})(h_j)) \\ \vdots \\ \Sigma_{j=1}^n(\alpha T_{n,j} + \beta S_{n,j})(h_j)) \end{pmatrix}
$$

$$\begin{aligned}
&= \begin{pmatrix} \Sigma_{j=1}^{n}(\alpha T_{1,j}(h_j) + \beta S_{1,j}(h_j)) \\ \vdots \\ \Sigma_{j=1}^{n}(\alpha T_{n,j}(h_j) + \beta S_{n,j}(h_j)) \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_{j=1}^{n}\alpha T_{1,j}(h_j) + \Sigma_{j=1}^{n}\beta S_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n}\alpha T_{n,j}(h_j) + \Sigma_{j=1}^{n}\beta S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha \begin{pmatrix} \Sigma_{j=1}^{n} T_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n} T_{n,j}(h_j) \end{pmatrix} + \beta \begin{pmatrix} \Sigma_{j=1}^{n} S_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n} S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha\phi\left([T_{i,j}]\right)(h) + \beta\phi([S_{i,j}])(h) \\
&= (\alpha\phi([T_{i,j}]) + \beta\phi([S_{i,j}]))(h) \\
&= \begin{pmatrix} \Sigma_{j=1}^{n}\alpha T_{1,j}(h_j) + \Sigma_{j=1}^{n}\beta S_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n}\alpha T_{n,j}(h_j) + \Sigma_{j=1}^{n}\beta S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha \begin{pmatrix} \Sigma_{j=1}^{n} T_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n} T_{n,j}(h_j) \end{pmatrix} + \beta \begin{pmatrix} \Sigma_{j=1}^{n} S_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n} S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha\phi([T_{i,j}])(h) + \beta\phi([S_{i,j}])(h) \\
&= (\alpha\phi([T_{i,j}]) + \beta\phi([S_{i,j}]))(h)
\end{aligned}$$

Thus, ϕ is linear.

We now show that ϕ is a bijection. That is, it is injective and surjective.

That ϕ is **injective**, let $E_k : \mathcal{H} \to \mathcal{H}^{(n)}$ be a map defined by $E_k(h_k) =$ vector that has h_k for its $k-$th entry and is 0 elsewhere.

Now suppose that $\phi([T_{i,j}]) = 0$, then

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \phi([T_{i,j}]E_k(h_k)) = \begin{pmatrix} T_{1,k}h_k \\ \vdots \\ T_{n,k}h_k \end{pmatrix}; k = \{1, ..., n\}.$$

Thus, $T_{i,k}h_k = 0, \ \forall \ h_k \in \mathcal{H}$ and $\forall \ i, k = \{1, ..., n\}$. Hence $[T_{i,j}] = 0$, so that ϕ is injective.

We now show that ϕ is **surjective**.

Define a map

$$E_j^* : \mathcal{H}^{(n)} \to \mathcal{H}.$$

We first show that this map sends a vector in $\mathcal{H}^{(n)}$ to its j-th component.

From $E_k : \mathcal{H} \to \mathcal{H}^{(n)}$, we have $\sum_{k=1}^{n} E_k h_k = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$.

Let $h_j \in \mathcal{H}$, $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^{(n)}$. Then by the definition of adjoints,

$$\left\langle E_j^* \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, h_j \right\rangle = \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, E_j h_j \right\rangle = \langle h_j, h. \rangle$$

Thus, E_j^* is the map that sends $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ to h_j as required. We show that ϕ is surjective. To show this, it suffices to show that $\phi([T_{i,j}]) = T$ for any

$T \in B(\mathcal{H}^{(n)})$.

To show this, let $T_{i,j} = E_i^* T E_j$, and $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathcal{H}^{(n)}$.

Then

$$\begin{aligned}
\left\langle \phi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} \sum_{j=1}^n T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j} h_j \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \\
&= \sum_{i,j=1}^n \langle T_{i,j} h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle E_i^* T E_j h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle T E_j h_j, E_i f_i \rangle \\
&= \left\langle \sum_{j=1}^n T E_j h_j, \sum_{i=1}^n E_i f_i \right\rangle \\
&= \left\langle T \left(\sum_{j=1}^n E_j h_j \right), \sum_{i=1}^n E_i f_i \right\rangle \\
&= \left\langle T \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle .
\end{aligned}$$

Thus, $\langle \phi([T_{i,j}])h, f \rangle = \langle Th, f \rangle \quad \Rightarrow \quad \langle \phi([T_{i,j}])h, f \rangle - \langle Th, f \rangle = 0 \quad \Rightarrow$ $\langle (\phi([T_{i,j}]) - T)h, f \rangle = 0 \ \Rightarrow \ \phi([T_{i,j}]) - T = 0$, thus $\phi([T_{i,j}]) = T$ And in addition since $\mathcal{H}$ is a Hilbert space, it follows that $\phi([T_{i,j}]) = T$, so that ϕ is surjective.

We now, show that ϕ is a **homomorphism**. Let $[S_{i,j}], [T_{i,j}] \in M_n(B(\mathcal{H}))$.

Then

$$\phi([S_{i,j}][T_{i,j}])(h) = \begin{pmatrix} \Sigma_{j=1}^{n}(S_{i,j}.T_{i,j})(h_j) \\ \vdots \\ \Sigma_{j=1}^{n}(S_{i,j}.T_{i,j})(h_j) \end{pmatrix} = \begin{pmatrix} \Sigma_{k=1}^{n}\Sigma_{j=1}^{n}S_{1,k}T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k=1}^{n}\Sigma_{j=1}^{n}S_{n,k}T_{k,j}(h_j) \end{pmatrix}$$
$$= \begin{pmatrix} \Sigma_{k,j=1}^{n}S_{1,k}T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k,j=1}^{n}S_{n,k}T_{k,j}(h_j) \end{pmatrix}.$$

So,

$$\phi([S_{i,j}][T_{i,j}])(h) = \begin{pmatrix} \Sigma_{k,j=1}^{n}S_{1,k}T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k,j=1}^{n}S_{n,k}T_{k,j}(h_j) \end{pmatrix}. \tag{1.5}$$

Also,

$$\begin{aligned} (\phi([S_{i,k}])\phi([T_{k,j}]))(h) &= \left(\phi([S_{i,k}]) \begin{pmatrix} \Sigma_{j=1}^{n}T_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^{n}T_{n,j}(h_j) \end{pmatrix}\right) \\ &= \begin{pmatrix} \Sigma_{k=1}^{n}S_{1,k}(\Sigma_{j=1}^{n}T_{k,j}(h_j)) \\ \vdots \\ \Sigma_{k=1}^{n}S_{n,k}(\Sigma_{j=1}^{n}T_{k,j}(h_j)) \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_{k=1}^{n}\Sigma_{j=1}^{n}S_{1,k}T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k=1}^{n}\Sigma_{j=1}^{n}S_{n,k}T_{k,j}(h_j) \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_{k,j=1}^{n}S_{1,k}T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k,j=1}^{n}S_{n,k}T_{k,j}(h_j) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \Sigma_{k=1}^n \Sigma_{j=1}^n S_{1,k} T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k=1}^n \Sigma_{j=1}^n S_{n,k} T_{k,j}(h_j) \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{k,j=1}^n S_{1,k} T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k,j=1}^n S_{n,k} T_{k,j}(h_j) \end{pmatrix}$$

Thus,

$$(\phi([S_{i,k}])\phi([T_{k,j}]))(h) = \begin{pmatrix} \Sigma_{k,j=1}^n S_{1,k} T_{k,j}(h_j) \\ \vdots \\ \Sigma_{k,j=1}^n S_{n,k} T_{k,j}(h_j) \end{pmatrix}. \tag{1.6}$$

Thus, from the equations (1.5) and (1.6), we have $\phi([S_{i,k}][T_{k,j}]) = \phi([S_{i,k}])\phi([T_{k,j}])$, so that ϕ is a homomorphism.

Next, we show that $\phi([T_{i,j}])^* = \phi([T_{j,i}^*])$, i.e. that ϕ is a $*$-**homomorphism**. Let $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathcal{H}^{(n)}$. Then

$$\begin{aligned} \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \phi([T_{i,j}])^* \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle &= \left\langle \phi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \Sigma_{j=1}^n T_{1,j}(h_j) \\ \vdots \\ \Sigma_{j=1}^n T_{n,j}(h_j) \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \\ &= \left\langle \Sigma_{j=1}^n T_{1,j} h_j, f_1 \right\rangle + ... + \left\langle \Sigma_{j=1}^n T_{n,j} h_j, f_n \right\rangle \\ &= \Sigma_{i=1}^n \Sigma_{j=1}^n \langle T_{i,j} h_j, f_i \rangle \end{aligned}$$

$$
\begin{aligned}
&= \Sigma_{i,j=1}^{n}\langle T_{i,j}h_j, f_i\rangle \\
&= \Sigma_{i,j=1}^{n}\langle h_j, T_{j,i}^{*}f_i\rangle \\
&= \left\langle h_1, \Sigma_{i=1}^{n}T_{1,i}^{*}f_i\right\rangle + ... + \left\langle h_n, \Sigma_{i=1}^{n}T_{n,i}^{*}f_i\right\rangle \\
&= \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} \Sigma_{i=1}^{n}T_{1,i}^{*}f_i \\ \vdots \\ \Sigma_{i=1}^{n}T_{n,i}^{*}f_i \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \phi([T_{1,i}^{*}]) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle.
\end{aligned}
$$

Thus, $\langle h, \phi([T_{i,j}])^{*}f\rangle = \langle h, \phi([T_{j,i}^{*}])f\rangle$

$\Rightarrow \langle h, \phi([T_{i,j}])^{*}f\rangle - \langle h, \phi([T_{j,i}^{*}])f\rangle = 0$

$\Rightarrow \langle h, (\phi([T_{i,j}])^{*} - \phi([T_{j,i}^{*}]))f\rangle = 0$

$\Rightarrow \phi([T_{i,j}])^{*} - \phi([T_{j,i}^{*}]) = 0$

$\Rightarrow \phi([T_{i,j}])^{*} = \phi([T_{j,i}^{*}])$. Thus ϕ is a $*-$homomorphism.

and hence a $*-$isomorphism. Moreover, this ϕ is a representation of $M_n(B(\mathcal{H}))$ on the Hilbert space $\mathcal{H}^{(n)}$. ϕ is called $*$-isomorphism of $M_n(B(\mathcal{H}))$ onto $B(\mathcal{H}^{(n)})$. Therefore, we can identify $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^{(n)})$. Thus, $M_n(B(\mathcal{H})) \cong B(\mathcal{H}^{(n)})$. This identification gives us a norm that makes the $*$-algebra $M_n(B(\mathcal{H}))$ a C^{*}-algebra by the proposition 1.7.4 below.

Proposition 1.7.4

Let $\phi : M_n(B(\mathcal{H})) \to B(\mathcal{H}^{(n)})$ be a $*$-isomorphism. Then

$$\|[T_{i,j}]\| = \|\phi([T_{i,j}])\|$$

is a norm and hence $M_n(B(\mathcal{H}))$ is a C^{*}-algebra.

PROOF. It is clear that $\|[T_{i,j}]\|$ is a norm and that it is submultiplicative. We now show the condition $\|[T_{i,j}]^*[T_{i,j}]\| = \|[T_{i,j}]\|^2, \ \forall \ [T_{i,j}] \in M_n(B(\mathcal{H}))$.

$$\begin{aligned}
\|[T_{i,j}]^*[T_{i,j}]\| &= \sup\{|\langle([T_{i,j}]^*[T_{i,j}])h, h\rangle| : h \in \mathcal{H}, \|h\| = 1\} \\
&= \sup\{|\langle([T_{i,j}])h, [T_{i,j}]h\rangle| : h \in \mathcal{H}, \|h\| = 1\} \\
&= \sup\{\|([T_{i,j}])h\|^2 : h \in \mathcal{H}, \|h\| = 1\} \\
&= \|[T_{i,j}]\|^2
\end{aligned}$$

We have, $\|[T_{i,j}]\|^2 = \|[T_{i,j}]^*[T_{i,j}]\|$. Hence $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^{(n)})$ are $*$-isomorphic $*$-algebras via $[T_{ij}] \leftrightarrow [T_{ij}]$. This means that $M_n(B(\mathcal{H}))$ is a C^*-algebra if we define the norm on it by considering the elements as operators on $\mathcal{H}^{(n)}$.

We also have that

$$\begin{aligned}
\|\phi([T_{i,j}]^*[T_{i,j}])\| &= \|\phi([T_{i,j}]^*)\phi([T_{i,j}])\| \\
&\leq \|\phi([T_{i,j}]^*)\|\|\phi([T_{i,j}])\| \\
&\leq \|\phi\|\|([T_{i,j}]^*)\|\|\phi\|\|([T_{i,j}])\| \\
&\leq \|([T_{i,j}]^*)\|\|([T_{i,j}])\| \\
&= \|[T_{i,j}]\|^2
\end{aligned}$$

since $M_n(B(\mathcal{H}))$ is a $*$-algebra. Hence

$$\|\phi([T_{i,j}]^*[T_{i,j}])\| \leq \|[T_{i,j}]\|^2. \tag{1.7}$$

It remains to show $\|\phi([T_{i,j}]^*[T_{i,j}])\| \geq \|[T_{i,j}]\|^2$. Let $h \in \mathcal{H}^{(n)}$ and

$\|h\| = 1$, then

$$\begin{aligned}
\|[T_{i,j}]\|^2 &= \|\phi([T_{i,j}])\|^2 \\
&= \sup\{\langle \phi([T_{i,j}])h, \phi([T_{i,j}])h\rangle : \|h\| = 1\} \\
&= \sup\{\langle \phi([T_{i,j}])^*\phi([T_{i,j}])h, h\rangle : \|h\| = 1\} \\
&= \sup\{\langle \phi([T_{i,j}]^*[T_{i,j}])h, h\rangle : \|h\| = 1\} \\
&= \sup\{|\langle \phi([T_{i,j}]^*[T_{i,j}])h, h\rangle| : \|h\| = 1\} \\
&\leq \sup\{\|\phi([T_{i,j}]^*[T_{i,j}])h\|\|h\| : \|h\| = 1\} \\
&\leq \sup\{\|\phi([T_{i,j}]^*[T_{i,j}])\|\|h\|^2 : \|h\| = 1\} \\
&\leq \|\phi([T_{i,j}]^*[T_{i,j}])\|.
\end{aligned}$$

Hence

$$\|[T_{i,j}]\|^2 \leq \|\phi([T_{i,j}]^*[T_{i,j}])\|. \tag{1.8}$$

From the inequalities (1.7) and (1.8), we have

$$\|\phi([T_{i,j}]^*[T_{i,j}])\| = \|[T_{i,j}]\|^2.$$

□

Proposition 1.7.5

If $\|.\|$ is a complete C^*-norm on a $*-$algebra $\mathcal{A}$, then it is given by the expression

$$\|a\| = r(a^*a)^{\frac{1}{2}},\ \forall\ a\ \in \mathcal{A},$$

where $r(a)$ is the spectral radius of a. Hence a C^*-norm on a $*-$algebra is unique if it exists

Theorem 1.7.6

There is a unique norm on $M_n(B(\mathcal{H}))$ making it a C^*-algebra.

Proof. That $M_n(B(\mathcal{H}))$ is a C^*-algebra has been shown in Proposition 1.7.4. It remains to show the uniqueness of this norm. For if $\|.\|_1$ and $\|.\|_2$ are norms on the $*-$algebra $M_n(B(\mathcal{H}))$ making it a C^*-algebra, then

$$\|[T_{i,j}]\|_1^2 = \|[T_{i,j}]^*[T_{i,j}]\|_1 = r([T_{i,j}]^*[T_{i,j}]) = \sup_{\lambda \in \sigma([T_{i,j}]^*[T_{i,j}])} |\lambda|^2.$$

Similarly,

$$\|[T_{i,j}]\|_2^2 = \|[T_{i,j}]^*[T_{i,j}]\|_2 = r([T_{i,j}]^*[T_{i,j}]) = \sup_{\lambda \in \sigma([T_{i,j}]^*[T_{i,j}])} |\lambda|^2.$$

Thus, $\|[T_{i,j}]\|_1 = \|[T_{i,j}]\|_2$. Hence the norm is unique. □

In conclusion, given an arbitrary C^*-algebra $\mathcal{A}$, by Gelfand Naimark Segal, $\mathcal{A}$ is a closed self-adjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This means that $M_n(\mathcal{A})$ is a closed self-adjoint subalgebra of the C^*-algebra $M_n(B(\mathcal{H})) \cong B(\mathcal{H}^{(n)})$ and hence a C^*-algebra.

Chapter 2

COMPLETELY POSITIVE MAPS

In this chapter, the notion of complete positivity has been introduced. Examples of positive and completely positive maps are discussed.

2.1 Properties of Positive Maps and operator systems

Definition 2.1.1

A linear map $\phi : M_n \to M_k$ is called positive **semidefinite** if for any real number l, matrices $A_1, ..., A_l \in M_n$, and any vectors $x_1, ..., x_l \in \mathbb{C}^k$, we have

$$\sum_{i,j=1}^{l} \langle \phi(A_i^* A_j) x_j, x_i \rangle . \geq 0,$$

or equivalently if it satisfies $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$.

Definition 2.1.2

A matrix $A \in M_n$ is called **positive** if it is Hermitian and all its eigenvalues are non-negative, or if there exists some matrix B such that it can be written $A = B^*B$, or equivalently if it is positive-semidefinite or if eigenvalues of A are nonnegative.

Definition 2.1.3

A linear map $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be **positive** if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. That is, if $\phi(\mathcal{A}^+) \subseteq \phi(\mathcal{B}^+)$.

This map is **bounded** if there exists an element $x \in \mathcal{A}$ such that $\|\phi(x)\| \leq c\|x\|, c \in \mathbb{C}$.

ϕ is **contractive** if $\|\phi\| \leq 1$

Definition 2.1.4

An **operator space** is any subspace $\mathcal{M}$ of some C^*-algebra, in particular, $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, together with a well-defined sequence of matrix norms on $M_n(\mathcal{M})$ where $M_n(\mathcal{M}) \subseteq M_n(B(\mathcal{H}))$ for all $n \geq 1$.

Thus an operator space carries not just an inherited norm structure, but these additional matrix norms, see Proposition 1.7.4

Definition 2.1.5

If $\mathcal{A}$ is a unital C^*-algebra, then a subspace $\mathcal{S} \subseteq \mathcal{A}$ such that $1 \in \mathcal{S}$ and $\mathcal{S} = \mathcal{S}^*$ is called an **operator system**, where

$$\mathcal{S}^* = \{a^* : a \in \mathcal{S}\}.$$

The following proposition shows that a positive map must be continuous.

Proposition 2.1.6

Let $\mathcal{S}$ be an operator system and $\mathcal{B}$ be a C^*-algebra with unit. If $\phi : \mathcal{S} \to \mathcal{B}$ is positive then $\|\phi(1)\| \leq \|\phi\| \leq 2\|\phi(1)\|$.

Proof. Let p be positive element in $\mathcal{S}$. Then $0 \leq p \leq \|p\|1$. By using linearity of ϕ we obtain that $0 \leq \phi(p) \leq \|p\|\phi(1)$. So $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$.

Any selfadjoint element $h \in \mathcal{S}$ is the difference of two positive elements in $\mathcal{S}$ since

$$h = \frac{\|h\|.1 + h}{2} - \frac{\|h\|.1 - h}{2}.$$

Again by linearity

$$\phi(h) = \phi\left(\frac{\|h\|.1 + h}{2}\right) - \phi\left(\frac{\|h\|.1 - h}{2}\right).$$

So $\phi(h)$ is the difference of two positive elements. By the first part of the proof we see that

$$\|\phi(h)\| \leq \frac{1}{2}\max\|\phi\left(\|h\|.1 + h\right)\|, \|\phi\left(\|h\|.1 - h\right)\| \leq \|h\|\|\phi(1)\|.$$

Finally, let $a \in \mathcal{S}$ be an arbitrary element. We can write $a = b + ic$ where b and c are selfadjoint with $\|b\|, \|c\| \leq \|a\|$.

Hence

$$\|\phi(a)\| \leq \|\phi(b)\| + \|\phi(c)\| \leq \|b\|\|\phi(1)\| + \|c\|\|\phi(1)\| \leq 2\|a\|\|\phi(1)\|.$$

This shows that $\|\phi\| \leq 2\|\phi(1)\|$. Since the other inequality is trivial, i.e $\|\phi(1)\| \leq \|\phi\|\|1\| = \|\phi\|.1 = \|\phi\|$, the assertion follows. $\square$

Theorem 2.1.7

Let $\mathcal{S}$ be an operator system and $\mathcal{B}$ be a C^*-algebra with unit. If $\phi : \mathcal{S} \to \mathcal{B}$ is a positive, linear functional on an operator system, then

$$\|\phi\| = \phi(1).$$

Proof. $0 \leq \|\phi(1)\| \leq \|\phi\|\|1\| \leq \|\phi\|$. Set $\phi(1) = 0$ Then,

$$\phi(1) \leq \|\phi\|. \tag{2.1}$$

On the other hand, let $a \in \mathcal{S}$ be positive. Then, $0 \leq a \leq \|a\|.1$.
Since ϕ is linear, we have

$$0 \leq \phi(a) \leq \|a\|.\phi(1).$$

Taking the supremum with $\|a\| \leq 1$, we get

$$\|\phi\| \leq \phi(1). \tag{2.2}$$

From inequalities (2.1) and (2.2) we obtain, $\|\phi\| = \phi(1)$.

$\square$

2 is the best constant in Proposition 2.1.6 as illustrated in the following example.

Example 2.1.8

Let $\mathcal{T}$ denote the unit circle in $\mathbb{C}$, $C(\mathcal{T})$ the space of continuous functions on $\mathcal{T}$, z the coordinate function, and $\mathcal{S} \subseteq C(\mathcal{T})$ the subspace spanned by

$1, z, \bar{z}$. Define $\phi : \mathcal{S} \longrightarrow \mathcal{B}$ by

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

$a + bz + c\bar{z} \in \mathcal{S}$ is positive if and only if $c = \bar{b}$ and $a \geq 2|b|$. A self-adjoint element of $M_2(\mathbb{C})$ is positive if and only if its diagonal entries and its determinant are nonnegative real numbers. Thus ϕ is positive. However

$$2\|\phi(1)\| = 2 = \|\phi(z)\| \leq \|\phi\| \Rightarrow \|\phi\| = 2\|\phi(1)\|.$$

We now consider properties of the domain that ensure that unital, positive maps are contractive.

Lemma 2.1.9

Let $\mathcal{A}$ be a C^*-algebra with a unit, and let $p_i, i = 1, ..., n,$ be positive elements of $\mathcal{A}$ such that

$$\sum_{i=1}^{n} p_i \leq 1.$$

If $\lambda_i, i = 1, ..., n,$ are scalars with $|\lambda_i| \leq 1$, then

$$\left\| \sum_{i=1}^{n} \lambda_i p_i \right\| \leq 1.$$

Proof. We have

$$\begin{pmatrix} \sum_{i=1}^{n} \lambda_i p_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} p_1^{1/2} & \cdots & p_1^{1/2} \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} \times \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \cdot \begin{pmatrix} p_1^{1/2} & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \\ p_1^{1/2} & \cdots & 0 \end{pmatrix}.$$

The norm of the matrix on the left is $\|\sum_{i=1}^n \lambda_i p_i\|$, while each of the three matrices on the right have norms less than 1.
Thus, $\| \sum_{i=1}^n \lambda_i p_i \| \leq 1$. □

Theorem 2.1.10

Let $\mathcal{B}$ be a C^*-algebra with unit, Ω be a compact Hausdorff space and $C(\Omega)$ be the space of continuous functions on Ω. Let $\phi : C(\Omega) \to \mathcal{B}$ be a positive map, then $\|\phi\| = \|\phi(1)\|$.

Proof. Assume that $\phi(1) \leq 1$. Let $f \in C(\Omega), \|f\| \leq 1$ and $\varepsilon > 0$ be given. First, choose a finite open covering $\{U_i\}_{i=1}^n$ of Ω such that $|f(\omega) - f(\omega_i)| < \varepsilon$ for $\omega \in U_i$, and let $\{p_i\}$ be a partition of unity subordinate to the covering. I.e, $\{p_i\}$ are nonnegative continuous functions satisfying $\sum_{i=1}^n p_i = 1$ and $p_i(x) = 0$ for $x \notin U_i, i = 1, ..., n$. Set $\lambda_i = f(\omega_i)$, and if $p_i(\omega) \neq 0$ for some i, then $\omega \in U_i$ and so $|f(\omega) - \lambda_i| < \varepsilon$. Hence for any ω,

$$\begin{aligned} \left| f(\omega) - \sum_{i=1}^n \lambda_i p_i(\omega) \right| &= \left| \sum_{i=1}^n (f(\omega) - \lambda_i) p_i(\omega) \right| \\ &\leq \sum_{i=1}^n |f(\omega) - \lambda_i| \, p_i(\omega) < \sum_{i=1}^n \varepsilon . p_i(\omega) = \varepsilon. \end{aligned}$$

Finally by Lemma 2.1.9, $\|\sum_{i=1}^{n} \lambda_i \phi(p_i)\| \leq 1$, so that

$$\|\phi(f)\| \leq \left\| \phi(f - \sum_{i=1}^{n} \lambda_i p_i) \right\| + \left\| \sum_{i=1}^{n} \lambda_i \phi(p_i) \right\| < 1 + \varepsilon . \|\phi\|,$$

and since ε was arbitrary, $\|\phi\| \leq 1$. □

Lemma 2.1.11

*(**Fejer-Riesz**). Let $\tau(e^{i\theta}) = \sum_{n=-N}^{+N} a_n e^{in\theta}$ be a strictly positive function on the unit circle T. Then there is a polynomial $p(z) = \sum_{n=0}^{N} p_n z^n$ such that*

$$\tau(e^{i\theta}) = |p(e^{i\theta})|^2.$$

Proof. Since τ is real-valued, $a_{-n} = \bar{a}_n$ and a_0 is real. Assume $a_{-N} \neq 0$. Set

$$g(z) = \sum_{n=-N}^{+N} a_n z^{n+N},$$

so that g is a polynomial of degree $2N$ with $g(0) \neq 0$. We have $g(e^{i\theta}) = \tau(e^{i\theta}).e^{iN\theta} \neq 0$. Antisymmetry of the coefficients of g implies

$$\overline{g(1/\bar{z})} = z^{-2N} g(z).$$

This means that $2N$ zeros of g may be written as $z_1, ..., z_N, 1/\bar{z}_1, ..., 1/\bar{z}_N$. Set $q(z) = (z - z_1)...(z - z_N), h(z) = (z - 1/\bar{z}_1)...(z - 1/\bar{z}_N)$ and have that

$$g(z) = a_N q(z) h(z),$$

with

$$\overline{h(z)} = \frac{(-1)^N \bar{z}^N q(1/\bar{z})}{z_1 ... z_N}.$$

Thus

$$\begin{aligned}\tau(e^{i\theta}) &= e^{-iN\theta}g(e^{i\theta}) = |g(e^{i\theta})| \\ &= |a_N|.|q(e^{i\theta})|.|\overline{h(e^{i\theta})}| \\ &= |\frac{a_N}{z_1...z_N}|.|q(e^{i\theta})|^2,\end{aligned}$$

so that $\tau(e^{i\theta}) = |p(e^{i\theta})|^2$, where $p(z) = |\frac{a_N}{z_1...z_N}|^{1/2}q(z)$. □

Writing $p(z) = \alpha_0 + ... + \alpha_N z^N$, then $\tau(e^{i\theta}) = \sum_{l,k=0}^{N} \alpha_l \bar{\alpha}_k e^{i(l-k)\theta}$, so that the coefficients of every strictly positive trigonometric polynomial (positive trigonometric polynomial) have this special form.

Theorem 2.1.12

Let T be an operator on a Hilbert space $\mathcal{H}$ with $\|T\| \leq 1$, and let $\mathcal{S} \subseteq C(T)$ be the operator system defined by

$$\mathcal{S} = \{p(e^{i\theta}) + \overline{q(e^{i\theta})} : p, q \;\; are \;\; polynomials\}.$$

Then the map $\phi : \mathcal{S} \longrightarrow B(\mathcal{H})$ defined by $\phi(p + \bar{q}) = p(T) + q(T)^*$ is positive.

Proof. We prove that $\phi(\tau)$ is positive for every strictly positive τ. Indeed, if τ is only positive, then $\tau + \varepsilon I$ is strictly positive for every $\varepsilon > 0$, and hence we have $\phi(\tau) + \varepsilon I = \phi(\tau + \varepsilon I) \geq 0$, it follows that $\phi(\tau) \geq 0$. Let $\tau(e^{i\theta})$ be strictly positive in $\mathcal{S}$, so that $\tau(e^{i\theta}) = \sum_{l,k=0}^{+n} \alpha_l \bar{\alpha}_k e^{i(l-k)\theta}$. We must prove that

$$\phi(\tau) = \sum_{l,k=0}^{+n} \alpha_l \bar{\alpha}_k T(l-k)$$

is a positive operator, where we define

$$T(j) = \begin{cases} T^j, & j \geq 0; \\ T^{*-j}, & j < 0. \end{cases}$$ Fix a vector x in $\mathcal{H}$ and note that

$$\langle \phi(\tau)x, x\rangle = \left\langle \begin{pmatrix} I & T^* & \cdots & T^{*n} \\ T & & \cdots & \\ \vdots & & & \\ . & & \cdots & T^* \\ T^n & \cdots & T & I \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 x \\ \vdots \\ \bar{\alpha}_n x \end{pmatrix}, \begin{pmatrix} \bar{\alpha}_1 x \\ \vdots \\ \bar{\alpha}_n x \end{pmatrix} \right\rangle. \quad (2.3)$$

Thus, we just show that the matrix operator is positive. Set

$$R = \begin{pmatrix} 0 & \cdots & & & 0 \\ T & \cdots & & & \\ 0 & \cdots & & & \\ \vdots & & & & \\ 0 & \cdots & 0 & T & 0 \end{pmatrix}$$

and note that $R^{n+1} = 0, \|R\| < 1$. Using I to denote the identity operator on $\mathcal{H}^{(n)}$, then the matrix operator in 2.3 can be written as

$$I + R + R^2 + ... + R^n + R^* + ... + R^{*n} = (I - R)^{-1} + (I - R^*)^{-1} - I.$$

This latter operator is positive, for, fix $h \in \mathcal{H}^{(n)}$, and let $h = (I - R)y$ for $y \in \mathcal{H}^{(n)}$. We obtain

$$\left\langle ((I - R)^{-1} + (I - R^*)^{-1} - I)h, h \right\rangle \ = \ \langle y, (I - R)y\rangle +$$

$$\begin{aligned} & \langle (I-R)y, y\rangle - \langle (I-R)y, (I-R)y\rangle \\ = \; & \|y\|^2 + \|Ry\|^2 \geq 0, \end{aligned}$$

since R is a contraction.

□

Corollary 2.1.13

Let $\mathcal{B}, \mathcal{C}$ be C^*-algebras with unit, let $\mathcal{A}$ be a subalgebra, $1 \in \mathcal{A}$, and let $\mathcal{S} = \mathcal{A} + \mathcal{A}^*$. If $\phi : \mathcal{S} \longrightarrow \mathcal{C}$ is positive, then $\|\phi(a)\| \leq \|\phi(1)\|.\|a\|$ for all $a \in \mathcal{A}$.

Proof. Let $a \in \mathcal{A}, \|a\| \leq 1$. By Proposition 2.1.6 and Example 2.1.8, we may extend ϕ to a positive map on the closure, $\overline{\mathcal{S}}$, of $\mathcal{S}$. There is a positive map $\psi : C(T) \longrightarrow \mathcal{B}$ with $\psi(p) = p(a)$. Since $\mathcal{A}$ is an algebra, the range of ψ is contained in $\overline{\mathcal{S}}$.
Clearly, the composition of positive maps is positive.

$$\|\phi(a)\| = \|\phi \circ \psi(e^{i\theta})\| \leq \|\phi \circ \psi(1)a\|.\|e^{i\theta}\| = \|\phi(1)a\| \leq \|\phi(1)\|\|a\|.$$

If $\|\phi(1)\| = 1$, then ϕ is a contraction. □

Corollary 2.1.14

(Russo-Dye). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit, and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a positive map. Then $\|\phi\| = \|\phi(1)\|$.*

Proof. By Corollary 2.1.13, we have $\|\phi(a)\| \leq \|\phi(1)\|.\|a\|$ for all $a \in \mathcal{A}$. Taking the supremum over all a with $\|a\| \leq 1$, we have

$$\|\phi\| \leq \|\phi(1)\|. \tag{2.4}$$

On the other hand, let $a = 1$ be the unit in $\mathcal{A}$ then from $\|\phi(a)\| \leq \|\phi(1)\|.\|a\|$ for all $a \in \mathcal{A}$, we have $\|\phi(a)\| \leq \|\phi(1)\|.\|a\| \Rightarrow \|\phi(1)\| \leq \|\phi(1)\|.\|1\| \leq \|\phi\|\|1\|\|1\| = \|\phi\|$. Thus,

$$\|\phi(1)\| \leq \|\phi\|. \tag{2.5}$$

From inequalities (2.4) and (2.5), we have $\|\phi\| = \|\phi(1)\|$. □

Proposition 2.1.15

Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a unital C^*-algebra and $\phi : \mathcal{S} \longrightarrow \mathcal{B}$ a unital contraction. Then ϕ is positive.

Proof. Define $f(a) = \langle \phi(a)x, x\rangle$, with $\|x\| = 1$. Then, we have

$$\begin{aligned}
\|f\| &= \sup |\langle \phi(a)x, x\rangle| : \|a\| \leq 1, \\
&= \sup_{\|a\|\leq 1} \|\phi(a)x\|\|x\| \\
&\leq \sup_{\|a\|\leq 1} \|\phi\|\|a\|\|x\|\|x\| \\
&= \|\phi\|.
\end{aligned}$$

Since $\|\phi\| \leq 1$, is a contraction, $\|f\| \leq 1$. Thus, $f > 0 \Rightarrow f(a) > 0$ so $f(a) = \langle \phi(a)x, x\rangle > 0$ for all $a > 0$, which implies that $\phi(a) > 0$ so that ϕ is positive. □

2.2 Examples of positive Maps

Example 2.2.1

Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a unital C^*-algebra and $\phi : \mathcal{S} \longrightarrow \mathcal{B}$ a unital contraction. Then ϕ is positive by proposition (2.1.15)

Example 2.2.2

Let $f : \mathcal{A} \to \mathbb{C}$ be a bounded linear functional on a C^*-algebra $\mathcal{A}$, then f is positive by theorem 1.6.24

Example 2.2.3

Let $\mathcal{A}$ be a C^*-algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$ a representation and $\phi : B(\mathcal{H}) \to \mathbb{C}$ a homomorphism defined by $\phi(a) = \langle \pi(a)h, h\rangle,\ a \in\ \mathcal{A},\ h \in\ \mathcal{H}$, then ϕ is positive.

For, let $a \in \mathcal{A}$ be positive, then there exists a $b \in \mathcal{A}$ such that $a = b^*b$.

$\phi(a^*a) = \langle \pi(a^*a)h, h\rangle = \langle \pi(a^*)\pi(a)h, h\rangle = \langle \pi(a)h, \pi(a)h\rangle\ =\ \|\pi(a)\|\ \geq\ 0.$

Therefore, $\phi(a)$ is positive and so is ϕ.

Example 2.2.4

Let $\mathcal{A}$ be a C^*-algebra and $\{E_{i,j}\}_{i,j=1}^{2} \in M_2(\mathcal{A})$ denote the system of matrix units for $M_n(\mathcal{A})$ with 1 at the i-row and j-column and zero elsewhere, i.e.
$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
Let $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be the transpose map, defined by

$$\phi(A) = A^T;\ \forall\ A \in \mathcal{A}.$$

Thus, $\phi(E_{i,j}) = E_{j,i}$.

Let $\mathcal{A} \to B(\mathcal{H})$ and $x \in \mathcal{H}$.

If $x = (-1, 2) \in \mathcal{H}$, we have

$$\langle E_{11}x, x\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle = 1.$$

We also have

$$\begin{aligned} \langle \phi(E_{11})x, x \rangle &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle = 1. \end{aligned}$$

Therefore, E_{11} and $\phi(E_{11})$ are positive. Hence a transpose map is positive.

2.3 Completely Positive Maps

If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, we endow $M_n(\mathcal{S})$ with the norm and order structure that it inherits as a subspace of $M_n(\mathcal{A})$.

Definition 2.3.1

Let $\mathcal{B}$ be a C^*-algebra, and $\phi : \mathcal{S} \longrightarrow \mathcal{B}$ a linear map, then we define maps $\phi_n : M_n(\mathcal{S}) \longrightarrow M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ for all $n \in \mathbb{N}$, $[a_{i,j}] \in M_n(\mathcal{S})$. ϕ is n-positive if ϕ_n is positive, and we call ϕ **completely positive** if ϕ is n-positive for all $n \in \mathbb{N}$, equivalently, if ϕ_n is positive for all $n \in \mathbb{N}$.

Proposition 2.3.2

Let $[T_{i,j}] \in M_n(B(\mathcal{H}))$, then $[T_{i,j}]$ is positive if and only if for every choice of n vectors $x_1, ..., x_n \in \mathcal{H}$, the scalar matrix $[\langle T_{i,j}x, x \rangle]$ is positive.

Proof. Let $[T_{i,j}] \in M_n(B(\mathcal{H}))$ be positive. We prove that, for every choice of n vectors $x_1, ..., x_n \in \mathcal{H}$, the scalar matrix $[\langle T_{i,j}x_j, x_i \rangle]$ is positive.

Since $[T_{i,j}] \in M_n(B(\mathcal{H}))$ is positive, there exists $[S_{i,j}] \in M_n(B(\mathcal{H}))$ such that

$$T_{i,j} = S_{j,i}^* S_{i,j}.$$

We have

$$\begin{aligned}
\langle T_{i,j} x_j, x_i \rangle &= \left\langle S_{j,i}^* S_{i,j} x_j, x_i \right\rangle \\
&= \langle S_{i,j} x_j, S_{i,j} x_i \rangle \\
&= \|S_{i,j} x\|^2 \geq 0.
\end{aligned}$$

Conversely, let the scalar matrix $[\langle T_{i,j} x_j, x_i \rangle]$ be positive for every $x \in B(\mathcal{H}^n)$, we prove that $[T_{i,j}] \in M_n(B(\mathcal{H}))$ is positive. This implies that

$$\langle T_{i,j} x_j, x_i \rangle = \langle x_i^* T_{i,j} x_j, 1 \rangle = \sum_{i,j=1}^{n} x_i^* T_{i,j} x_j \geq 0.$$

Let $\{\pi, \mathcal{H}, x_0\}$ be an arbitrary cyclic representation of $B(\mathcal{H})$. Then for each vector $x = \sum_{j=1}^{n} x_j \bigotimes e_j \in \tilde{\mathcal{H}} = \mathcal{H} \bigotimes \mathcal{H}_n$, we have

$$\begin{aligned}
(\tilde{\pi}(T)x|x) &= \sum_{i,j=1}^{n} \left(\pi(T)\left(x_j \bigotimes e_j\right)|x_i \bigotimes e_i\right) \\
&= \sum_{i,j,k=1}^{n} \left(\pi(T_{k,j})\left(x_j \bigotimes e_k\right)|x_i \bigotimes e_i\right) \\
&= \sum_{i,j=1}^{n} (\pi(T_{i,j})x_j|x_i).
\end{aligned}$$

Choose sequences $\{x_i^m : m = 1, 2, ...\} \in B(\mathcal{H})$ with $x_i = \lim_{m\to\infty} \pi(x_i^m)x_0$.

We have

$$\begin{aligned}(\tilde{\pi}(T)x|x) &= \lim_{m\to\infty}\sum_{i,j=1}^{n}\left(\pi(T_{i,j})\pi\left(x_j^m\right)x_0|\pi(x_i^m)x_0\right)\\ &= \lim_{m\to\infty}\left(\pi\left(\sum_{i,j=1}^{n}(x_i^m)^*T_{i,j}x_j^m\right)x_0|x_0\right)\geq 0.\end{aligned}$$

Hence $\tilde{\pi}(T)$ is positive for any cyclic representation π so that $T > 0$. Thus $[T_{i,j}]$ is positive.

□

Proposition 2.3.3

Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{A}$ be an operator space. We set $\mathcal{M}^* = \{a^* : a \in \mathcal{M}\}$, an operator space. If $\phi : \mathcal{M} \to B(\mathcal{H})$ is a linear map, then the map $\phi^* : \mathcal{M}^* \to B(\mathcal{H})^*$ defined by $\phi^*(a^*) = \phi(a^*)^*$ is also linear. We can define their corresponding linear maps: $\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$ by $\phi_n([a_{i,j}] = [\phi(a_{i,j})])$ where $[a_{i,j}] \in M_n(\mathcal{M})$ and $\phi_n^* : M_n(\mathcal{M}^*) \to M_n(B(\mathcal{H})^*)$ by $\phi_n^*([a_{i,j}]^*) = [\phi^*(a_{i,j})^*] = [\phi((a_{i,j})^*)^*]$, $\forall\, i,j = 1,...,n$, then $\phi^* = \phi$ and $\phi_n^* = \phi_n$.

Proof. Let $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{M}$, then by definition, $\alpha a + \beta b \in \mathcal{M}$ so that $(\alpha a + \beta b)^* \in \mathcal{M}^*$. We have

$$\begin{aligned}\phi^*\left((\alpha a+\beta b)^*\right) &= \phi\left((\alpha a+\beta b)^*\right)^*\\ &= \phi(\bar{\alpha}a^*+\bar{\beta}b^*)^*\\ &= \phi(\bar{\bar{\alpha}}(a^*)^*+\bar{\bar{\beta}}(b^*)^*)\\ &= (\bar{\bar{\alpha}}\phi(a^*)^*+\bar{\bar{\beta}}\phi(b^*)^*)\\ &= \alpha\phi^*(a^*)+\beta\phi^*(b^*)\end{aligned}$$

Hence ϕ^* is linear.

Let $h \in \mathcal{H}$, then

$$\begin{aligned}\langle[\phi^*(a_{i,j})^*]h, h\rangle &= \langle\phi_n^*([a_{i,j}])^*h, h\rangle \\ &= \langle\phi_n([a_{i,j}])^{**}h, h\rangle \\ &= \langle\phi_n([a_{i,j}])h, h\rangle \\ &= \langle[\phi(a_{i,j})]h, h\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}0 &= \langle[\phi^*(a_{i,j})^*]h, h\rangle - \langle[\phi(a_{i,j})]h, h\rangle \\ &= \langle[(\phi^* - \phi)(a_{i,j})]h, h\rangle \quad \textit{since } [a_{i,j}] \in M_n(\mathcal{A}) \textit{ is self adjoint.}\end{aligned}$$

Thus $\phi^* - \phi = 0$, so that $\phi^* = \phi$.

We also have,

$$\langle\phi_n^*([a_{i,j}]^*)h, h\rangle = \langle\phi_n([a_{i,j}])^{**}h, h\rangle = \langle\phi_n([a_{i,j}])h, h\rangle.$$

$$\begin{aligned}0 &= \langle[\phi_n^*(a_{i,j})^*]h, h\rangle - \langle[\phi_n(a_{i,j})]h, h\rangle \\ &= \langle[(\phi_n^* - \phi_n)(a_{i,j})]h, h\rangle \quad \textit{since } [a_{i,j}] \in M_n(\mathcal{A}) \textit{ is self adjoint.}\end{aligned}$$

Thus $\phi_n^* - \phi_n = 0$, so that $\phi_n^* = \phi_n$. Thus, ϕ_n is hermitian. $\square$

2.4 Examples Of Completely Positive Maps

Example 2.4.1

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism with $\pi(1) = 1$. Let also each of the maps $\pi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by
$\pi_n([a_{i,j}]) = [\pi(a_{i,j})]$ for all $[a_{i,j}] \in M_n(\mathcal{A})$ and for all $n \in \mathbb{N}$ be a $*$ -homomorphism, then π is completely positive.

Let $[a_{i,j}] \in M_n(\mathcal{A})$ be positive, then there exist $[b_{i,j}] \in M_n(\mathcal{A})$ such that
$[a_{i,j}] = [(b_{i,j})^* b_{i,j}]$.

We show that π is positive and completely positive. We have that

$$\pi(a_{i,j}) = \pi((b_{i,j})^* b_{i,j}) = \pi(b_{i,j})^* \pi(b_{i,j}) \geq 0.$$

This implies that π is positive.

Since each map, $\pi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by $\pi_n([a_{i,j}]) = [\pi(a_{i,j})]$ is a $*$-homomorphism, by definition we have

$$\begin{aligned} \pi_n([a_{i,j}]) = \pi_n([(b_{i,j})^* b_{i,j}]) &= [\pi((b_{i,j})^* b_{i,j})] \\ &= [\pi(b_{i,j})^* \pi(b_{i,j})] \geq 0. \end{aligned}$$

Thus, $\pi_n([a_{i,j}]) \geq 0$ for every $n \in \mathbb{N}$. This implies that π is completely positive.

Example 2.4.2

Let $\mathcal{A}$ be a C^*-algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$ a representation and $\phi \ : \ B(\mathcal{H}) \ \to \ \mathbb{C}$ a homomorphism defined by $\phi(a) = \langle \pi(a)h, h \rangle, \ a \in \mathcal{A}, \ h \in \mathcal{H}$. Let also

$\phi_n : M_n(B(\mathcal{H})) \to M_n(\mathbb{C})$ be linear maps defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$, for all $[a_{ij}] \in M_n(\mathcal{A})$, then ϕ is completely positive. For, let $a \in \mathcal{A}$ be positive, then

$$\phi(a^*a) = \langle \pi(a^*a)h, h\rangle = \langle \pi(a^*)\pi(a)h, h\rangle = \langle \pi(a)h, \pi(a)h\rangle \ = \ \|\pi(a)\| \ \geq \ 0.$$

Therefore, $\phi(a)$ is positive and so is ϕ.

Let $[a_{ij}]$ be positive and Let $h \in \mathcal{H}^n$, we have

$$\begin{aligned} \phi_n([a_{ij}]^*[a_{ij}]) = [\phi(a_{ij})^*(a_{ij})] &= \langle [\pi((a_{ij})^*(a_{ij}))]h, h\rangle \\ &= \langle [\pi(b_{ji}^*)\pi(b_{ij})]h, h\rangle \\ &= \langle [\pi(b_{ij})]h, \pi(b_{ij})h\rangle \\ &= \|[\pi(a_{ij})]h\|^2 \geq 0. \end{aligned}$$

Hence ϕ is completely positive.

Example 2.4.3

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and x and y in $\mathcal{A}$ be diagonal matrices, where $[x_{i,j}] = \begin{cases} x, & \text{i=j;} \\ 0, & i \neq j. \end{cases}$ and $[y_{i,j}] = \begin{cases} y, & \text{i=j;} \\ 0, & i \neq j. \end{cases}$

Define $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\phi(a) = xay$. Then $\phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{A})$ is given by $\phi_n([a_{i,j}]) = [x_{i,j}][a_{i,j}][y_{i,j}], \quad [x_{i,j}], [a_{i,j}], [y_{i,j}] \in M_n(\mathcal{A})$, then Let $x = y^*$ and we may assume without loss of generality that $(a_{i,j})$ is a unit matrix. Then,

$$\phi(a) \ = \ y^*ay$$

$$
\begin{aligned}
&= \begin{pmatrix} y^* & 0 & \cdots & 0 \\ 0 & \cdots & & \\ \vdots & & & \\ 0 & \cdots & 0 & y^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y \end{pmatrix} \\
&= \begin{pmatrix} y^* & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y^* \end{pmatrix} \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & \cdots & . \\ \vdots & & & \\ 0 & \cdots & 0 & y \end{pmatrix} = \begin{pmatrix} y^*y & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y^*y \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\phi(a) &= \begin{pmatrix} y^* & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y \end{pmatrix} \\
&= \begin{pmatrix} y^* & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y^* \end{pmatrix} \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y \end{pmatrix} = \begin{pmatrix} y^*y & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ 0 & \cdots & 0 & y^*y \end{pmatrix},
\end{aligned}
$$

which is positive.

Chapter 3

COMPLETELY BOUNDED MAPS

In this chapter, we have given the relationship of boundedness and complete boundedness. Some of the elementary properties of the completely bounded norm and relationship of complete positivity against complete boundedness have been investigated. Examples and counter examples have also been discussed.

Definition 3.0.4

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and $\phi : \mathcal{M} \to \mathcal{B}$ a linear map, then we define maps $\phi_n : M_n(\mathcal{M}) \to M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ for all $n \in \mathbb{N}$, $[a_{i,j}] \in M_n(\mathcal{M})$. ϕ is **completely bounded** if the completely bounded norm $\|\phi\|_{cb}$ is finite, that is, the $\sup\{\|\phi_n\| : n \in \mathbb{N}\}$ is finite. We set

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty.$$

Completely contractive indicates that $\|\phi\|_{cb} \leq 1$.

Proposition 3.0.5

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map from the C^*-algebra $\mathcal{A}$ to C^*-algebra $\mathcal{B}$. Let the maps $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$, $[a_{i,j}] \in M_n(\mathcal{A})$ be positive for all n, then $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\}$ is a norm on $[a_{i,j}] \in M_n(\mathcal{A})$.

PROOF. **(a)** We show that $\|\phi\|_{cb}$ is non-negative.

We have that

$$\|\phi_n\| = \sup_{n\in\mathbb{N}} \{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1\} = \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1\} \geq 0.$$

Therefore,

$$\begin{aligned} \|\phi\|_{cb} &= \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty \\ &= \sup \left\{ \sup_{n\in\mathbb{N}} \{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1\} \right\} \\ &\leq \sup \left\{ \sup_{n\in\mathbb{N}} \{\|\phi_n\| \|([a_{i,j}])\| : \|a_{i,j}\| \leq 1\} \right\} \\ &= \sup_{n\in\mathbb{N}} \|\phi_n\|. \end{aligned}$$

we have that

$$\|\phi\|_{cb} \leq \sup \|\phi_n\| < \infty,$$

for some $n \in \mathbb{N}$. That is, there is an $n \in \mathbb{N}$ such that

$$\|\phi\|_{cb} \leq \|\phi_n\|. \tag{3.1}$$

Also, the complete bound norm of ϕ is given by

$$\|\phi\|_{cb} = \sup_n \|\phi_n\| < \infty.$$

This implies that

$$\|\phi\|_{cb} \geq \|\phi_n\|, \ \forall\, n \in \mathbb{N}. \tag{3.2}$$

So from inequalities (3.1) and (3.2) we have

$$\|\phi\|_{cb} = \|\phi_n\| > 0. \tag{3.3}$$

Thus,

$$\|\phi\|_{cb} = \sup_n \|\phi_n\|$$

is non-negative.

(b) We prove that $\|\phi\|_{cb} = \sup_n \|\phi_n\| = 0$ if and only if $\phi_n = 0$.

If $\phi_n = 0, \ \forall\, n \in \mathbb{N}$, then $\|\phi_n\| = 0, \ \forall\, n \in \mathbb{N}$. Thus,

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\} = 0.$$

(c) Next, we show that $\|\alpha\phi\|_{cb} = |\alpha|\|\phi\|_{cb}, \ \alpha \in \mathbb{C}$.

We have,

$$\begin{aligned} \|\alpha\phi\|_{cb} = \sup\{\|\alpha\phi_n\| : n \in \mathbb{N}\} &= \sup_{n\in\mathbb{N}} |\alpha|\|\phi_n\| \\ &= |\alpha| \sup_{n\in\mathbb{N}} \|\phi_n\| \\ &= |\alpha|\|\phi\|_{cb}. \end{aligned}$$

(d) Finally, we show the triangle inequality.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ also be a linear map. Then, it is clear that $\phi + \varphi : \mathcal{A} \to \mathcal{B}$ is also linear. Let the maps $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ be defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$, for all $[a_{i,j}] \in M_n(\mathcal{A})$. Then,

$$\begin{aligned} \|\phi + \varphi\|_{cb} = \sup_{n\in\mathbb{N}} \|\phi_n + \varphi_n\| &\leq \sup_{n\in\mathbb{N}}\{\|\phi_n\| + \|\varphi_n\|\} \\ &\leq \sup_{n\in\mathbb{N}} \|\phi_n\| + \sup_{n\in\mathbb{N}} \|\varphi_n\| \\ &= \|\phi\|_{cb} + \|\varphi\|_{cb} \ \ (\textit{triangle inequality}). \end{aligned}$$

Thus, $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\}$ is indeed a norm on $[a_{i,j}] \in M_n(\mathcal{A})$.

□

If ϕ is n-positive, then ϕ is k-positive for $k \leq n$. Also $\|\phi_k\| \leq \|\phi_n\|$ for $k \leq n$.

To show the first part of this statement, suppose that ϕ is completely positive, then by definition ϕ is n-positive for all $n = 1, 2, ..., k, ... ,\ n - 1$ so that ϕ is k-positive. For the second part, we prove the following proposition.

Proposition 3.0.6

Given a C^*-algebra $\mathcal{A}$, an operator space $\mathcal{M} \subseteq \mathcal{A}$ and a linear map $\phi : \mathcal{M} \to \mathcal{B}$. Let the maps $\phi_n : M_n(\mathcal{M}) \to M_n(\mathcal{B})$ be defined by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$, $\forall\ [a_{i,j}] \in M_n(\mathcal{A})$, then norms $\|\phi_n\|$, $n \in \mathbb{N}$ form an increasing sequence

$$\|\phi\| \leq \|\phi_2\| \leq ... \leq \|\phi_n\|$$

and $$\|\phi_n\| \leq n\|\phi\|,\ \forall\, n \in \mathbb{N}.$$

PROOF. We define the linear map $\phi : \mathcal{M} \to B(\mathcal{H})$ by $\phi(a) = \langle \phi(a)x, x\rangle$, where $a \in \mathcal{M}$ and $x \in \mathcal{H}$

Let $n = 1$, then by definition $\phi_n([a_{i,j}]) = [\phi(a_{i,j})],\ i, j = 1$, we have $\phi_1([a_{11}]) = [\phi(a_{11})]$. Therefore

$$\begin{aligned} \|\phi_1\| &= \sup\{\|\phi_1([a_{i,j}])\| : \|a_{i,j}\| \leq 1,\ i,j = 1\} \\ &= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i, j = 1\} \\ &= \sup\{\|[\phi(a_{11})]\| : \|a_{1,1}\| \leq 1\} = \|\phi\|. \end{aligned}$$

ϕ_1 coincides with ϕ so that $\|\phi\| = \|\phi_1\|$.

We now consider case when $n = 2$.
Let $[a_{i,j}] \in M_2(\mathcal{M})\ \ i, j = 1, 2$, then for the maps $\phi_2 : M_2(\mathcal{M}) \to M_2(B(\mathcal{H}))$, defined by $\phi_2([a_{i,j}]) = [\phi(a_{i,j})],\ i, j = 1, 2$ where $\phi(a_{i,j}) = \langle \phi(a_{i,j})x_j, x_i\rangle$
We have

$$\begin{aligned} \|\phi_2\| &= \sup\{\|\phi_2([a_{i,j}])\| : \|a_{i,j}\| \leq 1, i, j = 1, 2\} \\ &= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i, j = 1, 2\} \\ &\geq \sup\{\|[\phi(a_{1,1})]\| : \|a_{1,1}\| \leq 1\} = \|\phi_1\|. \end{aligned}$$

Thus $\|\phi_2\| \geq \|\phi_1\|$.

We have

$$\begin{aligned} \|\phi_3\| &= \sup\{\|\phi_3([a_{i,j}])\| : \|a_{i,j}\| \leq 1, i, j = 1, 2, 3\} \\ &= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i, j = 1, 2, 3\} \\ &\geq \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i, j = 1, 2\} = \|\phi_2\|. \end{aligned}$$

Thus $\|\phi_3\| \geq \|\phi_2\|$.

In general, consider the maps

$$\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$$

defined by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})] \ \forall \ i,j = 1,2,...,n$. By the above calculation we have that

$$\begin{aligned} \|\phi_n\| &= \sup\{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1, i,j = 1,...,n\} \\ &= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i,j = 1,...,n\} \\ &\geq \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i,j = 1,...,n-1\} = \|\phi_{n-1}\|. \end{aligned}$$

Thus $\|\phi_n\| \geq \|\phi_{n-1}\|$.

Thus, by upward induction we have,

$$\|\phi\| \leq \|\phi_2\| \leq ... \leq \|\phi_{n-1}\| \leq \|\phi_n\|.$$

In the second part of the proposition, we show that $\|\phi_n\| \leq n\|\phi\|$. Let $\|a_{i,j}\| \leq 1$ for all $i,j = 1,...,n$. Define $\|[a_{i,j}]\|$ for all $[a_{i,j}] \in M_n(\mathcal{M})$ by

$$\|[a_{i,j}]\| = \sqrt{\sum_{i,j=1}^{n} \|a_{i,j}\|^2}.$$

Then we have,

$$\|\phi_n\| = \sup\{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1, i,j = 1,...,n; \ \ n \in \mathbb{N}\}$$

$$\begin{aligned}
&= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, i,j = 1,...,n\} \\
&= \sup\left\{\sqrt{\sum_{i,j=1}^{n} \|\phi(a_{i,j})\|^2} : \|a_{i,j}\| \leq 1, i,j = 1,...,n\right\} \\
&\leq \sup\left\{\sqrt{\sum_{i,j=1}^{n} \|\phi\|^2\|a_{i,j}\|^2} : \|a_{i,j}\| \leq 1, i,j = 1,...,n\right\} \\
&= \|\phi\| \sup\left\{\sqrt{\sum_{i,j=1}^{n} \|a_{i,j}\|^2} : \|a_{i,j}\| \leq 1, i,j = 1,...,n\right\} \\
&= n\|\phi\|.
\end{aligned}$$

Thus $\|\phi_n\| \leq n\|\phi\|$ for all $n \in \mathbb{N}$

□

3.1 Properties of Completely bounded Maps

Lemma 3.1.1

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism with $\pi(1) = 1$. Let also each of the maps $\pi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by $\pi_n([a_{i,j}]) = [\pi(a_{i,j})]$ for all $[a_{i,j}] \in M_n(\mathcal{A})$ and for all $n \in \mathbb{N}$ be a $*$-homomorphism, then π is completely positive and completely bounded and that $\|\pi\| = \|\pi_n\| = \|\pi\|_{cb} = 1$.

Proof. Complete positivity of the map has been shown in example 2.4.1.

We now prove the boundedness and complete boundedness of π. Let $\pi([a_{i,j}]) = [\pi(a_{i,j})] = [c_{i,j}] \in M_n(\mathcal{B})$ for some $[a_{i,j}] \in M_n(\mathcal{A})$. Then,

$$\|\pi\| = \sup\{\|\pi([a_{i,j}])\| : \|a_{i,j}\| \leq 1;\ n \in \mathbb{N}\}$$

$$
\begin{aligned}
&= \sup\{\|[\pi(a_{i,j})]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[c_{i,j}]\| : \|c_{i,j}\| \leq 1\} \\
&= 1.
\end{aligned}
$$

This implies that π is bounded. That is, a positive map is bounded.

Now, let $n \in \mathbb{N}$ be finite, then

$$
\begin{aligned}
\|\pi_n\| &= \sup\{\|\pi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1, i = 1, ..., n; j = 1, ..., m; \\
&\quad 1 \leq m \leq n \in \mathbb{N}; i \times j = n \times n\} \\
&= \sup\left\{\|[\pi(a_{i,j})]\| : \|a_{i,j}\| \leq 1,\ i = 1, ..., n;\ \ j = 1, ..., m\right\} \\
&= \sup\left\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\|\pi(a_{i,j})\|^2} : \|a_{i,j}\| \leq 1\right\} \\
&\leq \sup\left\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\|\pi\|^2\|a_{i,j}\|^2} : \|a_{i,j}\| \leq 1\right\} \\
&= \|\pi\| \sup\left\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\|a_{i,j}\|^2} : \|a_{i,j}\| \leq 1\right\} \\
&= n\|\pi\| = n.
\end{aligned}
$$

Thus $\|\pi_n\| \leq n$.

Taking the supremum of both sides, we get

$$
\sup\{\|\pi_n\| : n \in \mathbb{N}\} = n.
$$

Therefore,

$$\|\pi\|_{cb} = \sup\{\|\pi_n\| : n \in \mathbb{N}\} = n < \infty.$$

Thus π is completely bounded.

We now show that $\|\pi\| = \|\pi_n\| = \|\pi\|_{cb} = 1$ with $\pi(1) = 1$.

$$\begin{aligned}
\|\pi\| &= \sup\{\|\pi([a_{i,j}])\| : \|a_{i,j}\| \leq 1, \ \ n \in \mathbb{N}, \ i,j = 1, ..., n\}. \\
&= \sup\{\|[\pi(a_{i,j})]\| : \|a_{i,j}\| \leq 1 \ \ i,j = 1, ..., n\}, \ \pi \ is \ linear.
\end{aligned}$$

$$\begin{aligned}
\|\pi_n\| &= \sup\{\|\pi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1, \ \ n \in \mathbb{N}, \ i,j = 1, ..., n\}. \\
&= \sup\{\|[\pi(a_{i,j})]\| : \|a_{i,j}\| \leq 1, \ i,j = 1, ..., n\}. \\
&= \|\pi\|.
\end{aligned}$$

$\Rightarrow \|\pi_n\| = \|\pi\|$.

From equation (3.3), we have that $\|\pi_n\| = \|\pi\|_{cb}$.

We now let $a_{i,j} = 1$, then

$$\begin{aligned}
\|\pi_n\| &= \sup\{\|\pi_n([1])\| : n \in \mathbb{N}\}. \\
&= \sup\{\|[\pi(1)]\|\} \\
&= \sup\{\|[1]\|\} = 1.
\end{aligned}$$

$\Rightarrow \|\pi_n\| = 1$. Furthermore, taking the supremum of both sides of this equation, we have

$$\sup\{\|\pi_n\| : n \in \mathbb{N}\} = 1,$$

which implies that $\|\pi\|_{cb} = \sup\{\|\pi_n\| : n \in \mathbb{N}\} = 1 < \infty$. Thus, π is completely bounded and that,

$$\|\pi\| = \|\pi_n\| = \|\pi\|_{cb} = 1.$$

$\square$

Proposition 3.1.2

Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{A}$ be an operator space. We set $\mathcal{M}^* = \{a^* : a \in \mathcal{M}\}$, an operator space. If $\phi : \mathcal{M} \to B(\mathcal{H})$ is a linear map, then the map $\phi^* : \mathcal{M}^* \to B(\mathcal{H})^*$ defined by $\phi^*(a^*) = \phi(a^*)^*$ is also linear. We can define their corresponding linear maps: $\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$ by $\phi_n([a_{i,j}] = [\phi(a_{i,j})])$ where $[a_{i,j}] \in M_n(\mathcal{M})$ and $\phi_n^* : M_n(\mathcal{M}^*) \to M_n(B(\mathcal{H})^*)$ by $\phi_n^*([a_{i,j}]^*) = [\phi^*(a_{i,j})^*] = [\phi((a_{i,j})^*)^*]$, $\forall\ i,j = 1, ..., n$, then $\|\phi_n\| = \|\phi_n^*\|$ and hence $\|\phi_n\|_{cb} = \|\phi_n^*\|_{cb}$, for all $n \in \mathbb{N}$.

Proof. By definition,

$$\begin{aligned}
\|\phi_n^*\| &= \sup_{n\in\mathbb{N}} \{\|\phi_n^*([a_{i,j}]^*)\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[\phi(a_{j,i}^*)^*]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[\phi(a_{i,j}^{**})]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[\phi(a_{i,j})]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup_{n\in\mathbb{N}}\{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1\} \\
&\leq \sup_{n\in\mathbb{N}}\{\|\phi_n\|\|[a_{i,j}]\| : \|a_{i,j}\| \leq 1\} \\
&= \|\phi_n\|.
\end{aligned}$$

Thus,

$$\|\phi_n^*\| \leq \|\phi_n\|. \tag{3.4}$$

Similarly,

$$\begin{aligned}
\|\phi_n\| &= \sup_{n\in\mathbb{N}}\{\|\phi_n([a_{i,j}])\| : \|a_{i,j}\| \leq 1\} \\
&= \sup_{n\in\mathbb{N}}\{\|\phi_n([a_{i,j}^{**}])\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[\phi(a_{j,i}^*)^*]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup\{\|[\phi^*(a_{j,i}^*)]\| : \|a_{i,j}\| \leq 1\} \\
&= \sup_{n\in\mathbb{N}}\{\|\phi_n^*([a_{i,j}]^*)\| : \|a_{i,j}\| \leq 1\} \\
&\leq \sup_{n\in\mathbb{N}}\{\|\phi_n^*\|\|[a_{i,j}]^*\| : \|a_{i,j}\| \leq 1\} \\
&= \|\phi_n^*\|.
\end{aligned}$$

Thus,

$$\|\phi_n\| \leq \|\phi_n^*\|. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\|\phi_n\| = \|\phi_n^*\|. \tag{3.6}$$

Taking the supremum over n on both sides of equation (3.6), we get

$$\|\phi\|_{cb} = \sup_{n\in\mathbb{N}}\|\phi_n\| = \sup_{n\in\mathbb{N}}\|\phi_n^*\| = \|\phi^*\|_{cb}.$$

$\square$

3.2 Examples Of Completely Bounded Maps.

Example 3.2.1

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism, then ϕ is completely positive and completely bounded by Lemma 3.1.1.

Example 3.2.2

Let $\mathcal{A}$ be a C^*-algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$ a representation and $\phi\ :\ B(\mathcal{H})\ \to\ \mathbb{C}$ a homomorphism defined by $\phi(a) = \langle \pi(a)h, h\rangle,\ a \in \mathcal{A},\ h \in \mathcal{H}$. Let also $\phi_n : M_n(B(\mathcal{H})) \to M_n(\mathbb{C})$ be linear maps defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$, for all $[a_{ij}] \in M_n(\mathcal{A})$, then ϕ is completely bounded.

Let $\pi(a_{i,j}) = c_{i,j} \in M_n(B(\mathcal{H}))$. We have,

$$\begin{aligned}
\|\phi([a_{i,j}])\| &= \sup\{|\langle[\pi(a_{i,j})]h, h\rangle| : \|h\| \leq 1, \|a_{i,j}\| \leq 1\}\\
&= \sup\{\|[\pi(a_{i,j})]h\|\|h\| : \|h\| \leq 1, \|a_{i,j}\| \leq 1\}\\
&\leq \sup\{\|\pi(a_{i,j})\|\|h\|\|h\| : \|h\| \leq 1, \|a_{i,j}\| \leq 1\}\\
&= \sup\{\|c_{i,j}\|\|h\|^2 : \|h\| \leq 1, \|c_{i,j}\| \leq 1\}\\
&= 1.
\end{aligned}$$

That is,

$$\|\phi\| \leq 1 \tag{3.7}$$

Hence ϕ is contractive. This also implies that ϕ, is bounded.

Now,

$$\begin{aligned}
\|\phi_n([a_{ij}])\| &= \|[\phi(a_{ij})]\|\\
&= \sup\{|\langle[\pi(a_{ij})h, h]\rangle| : \|h\| \leq 1, \|a_{i,j}\| \leq 1\ \ i, j = 1, ..., n \in \mathbb{N}\}\\
&= \sup\{\|[\pi(a_{ij})]h\|\|h\| : \|h\| \leq 1, \|a_{i,j}\| \leq 1\ \ i, j = 1, ..., n\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup\{\|\pi(a_{ij})\|\|h\|\|h\| : \|h\| \leq 1, \|a_{i,j}\| \leq 1 \ \ i,j = 1,...,n\} \\
&\leq \sup\{\|\pi\|\|a_{ij}\|\|h\|^2 : \|h\| \leq 1, \|c_{i,j}\| \leq 1 \ \ i,j = 1,...,n\} \\
&= \|\pi\| = 1.
\end{aligned}$$

From this, $\|\phi_n\| \leq 1$. Thus, ϕ_n is bounded. Taking the supremum on both sides of this equation. That is, $\sup_{n\in\mathbb{N}} \|\pi_n\| = 1$, which implies that ϕ is completely bounded.

Let $a = 1$, then $\phi(1) = \langle \pi(1)h, h\rangle = \langle h, h\rangle = \|h\|^2 = 1$.

We also have that $\|\phi_n([a_{ij}])\| = \|[\phi(a_{ij})]\| \leq \|\phi\|\|(a_{ij})\|$ so that $\|\phi_n\| \ \leq \ \|\phi\|$.

On the other hand $\|\phi\| \ \leq \ \|\phi_n\|$ by the lemma above, so that the equality holds.

$1 = \phi(1)$. Taking the norm on both sides, we get
$1 \ = \ \|1\| \ = \ \|\phi(1)\| \ \leq \ \|\phi\|\|1\| \ = \ \|\phi\| \ \Rightarrow \ \|\phi\| \ \geq \ 1.$

Together, we have $\|\phi\| = 1$.

Example 3.2.3

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and x and y in $\mathcal{A}$ be diagonal matrices, where $[x_{i,j}] = \begin{cases} x, & \text{i=j;} \\ 0, & i \neq j. \end{cases}$ and $[y_{i,j}] = \begin{cases} y, & \text{i=j;} \\ 0, & i \neq j. \end{cases}$

Define $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\phi(a) = xay$. Then $\phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{A})$ is given by $\phi_n([a_{i,j}]) = [x_{i,j}][a_{i,j}][y_{i,j}], \ \ [x_{i,j}], [a_{i,j}], [y_{i,j}] \in M_n(\mathcal{A})$, then

$$\|\phi_n((a_{i,j}))\| \ = \ \|(xa_{i,j}y)\|$$

$$
\begin{aligned}
&= \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & x \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & y \end{pmatrix} \\
&\leq \|x\|\|(a_{i,j})\|\|y\|
\end{aligned}
$$

Taking the supremum over all $(a_{i,j})$ with $\|(a_{i,j})\| \leq 1$, we obtain

$$\|\phi_n\| \leq \|x\|\|y\|.$$

Thus, ϕ is completely bounded, and taking the supremum over all $n \in \mathbb{N}$,

$$\sup_n \|\phi_n\| = \|\phi\|_{cb} \leq \|x\|.\|y\|.$$

Example 3.2.4

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $V_i : \mathcal{H}_1 \longrightarrow \mathcal{H}_2, i = 1, 2$, be bounded operators and $\pi : \mathcal{A} \longrightarrow B(\mathcal{H}_2)$ be a $*$-homomorphism. Define $\phi : \mathcal{A} \to B(\mathcal{H}_1)$ by

$$\phi(a) = V_2^* \pi(a) V_1 \ \forall\ a \in \mathcal{A}.$$

Then ϕ is completely bounded and that

$$\|\phi\|_{cb} \leq \|V_1\|\|V_2\|.$$

Let $x, y \in \mathcal{H}_1$ be of unit lengths, then

$$
\begin{aligned}
|\langle \phi_n(a)x, y\rangle| &= |\langle V_2^* \otimes I_n \pi(a) V_1 \otimes I_n x, y\rangle| \\
&= |\langle \pi(a) V_1 \otimes I_n x, V_2 \otimes I_n y\rangle|
\end{aligned}
$$

$$\begin{aligned}
&\leq \|\pi(a)V_1 \otimes I_n x\| \|V_2 \otimes I_n y\| \\
&\leq \|\pi\| \|(a)\| \|V_1\| \|x\| \|V_2\| \|y\| \\
&\leq \|(a)\| \|V_1\| \|x\| \|V_2\| \|y\| \ since \ \|\pi\| = 1
\end{aligned}$$

$$\begin{aligned}
&\leq \|\pi(a)\| \|V_1 \otimes I_n\| \|x\| \|V_2 \otimes I_n\| \|y\| \\
&\leq \|\pi\| \|(a)\| \|V_1\| \|x\| \|V_2\| \|y\| \\
&\leq \|(a)\| \|V_1\| \|x\| \|V_2\| \|y\| \ since \ \|\pi\| = 1
\end{aligned}$$

Taking the supremum over all $n \in \mathbb{N}$ with $\|a\| \leq 1, \|x\| \leq 1, \|y\| \leq 1$, we have that $\sup_n \|\phi_n\| \leq \|V_1\|\|V_2\| < \infty$, since V_1 and V_2 are bounded. Thus ϕ is completely bounded and $\|\phi\|_{cb} \leq \|V_1\|\|V_2\|$.

Example 3.2.5

This is a counter example. That is, a map that is completely bounded but not completely positive.

Let $\mathcal{A}$ be a C^*-algebra and $\{E_{i,j}\}_{i,j=1}^2 \in M_2(\mathcal{A})$ denote the system of matrix units for $M_2(\mathcal{A})$ with 1 at the i-row and j-column and zero elsewhere, i.e.

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be the transpose map, defined by

$$\phi(A) = A^T; \ \forall \ A \in \mathcal{A}.$$

ϕ is positive by example 2.2.4

Also, $\|E_{i,j}\| = \sup\{\|E_{i,j}x\| : x \in \mathcal{H}, \ \|x\| = 1\} = 1$.

So $\|\phi(E_{i,j})\| = \|E_{j,i}\| = \sup\{\|E_{j,i}x\| : x \in \mathcal{H}, \ \|x\| = 1\} = 1$.

Taking supremum with $\|E_{i,j}\| \leq 1$, we have $\|\phi\| = 1$.

Alternatively, $\|E_{i,j}\| = \sup |\langle E_{i,j}x, x\rangle| = 1 < \infty$. Thus $E_{i,j}$ is bounded and

$\|\phi(E_{i,j})\| = \sup |\langle \phi(E_{i,j})x, x\rangle| = 1 < \infty$. Hence, $\phi = \phi_1$ is bounded.

Thus, the transpose of a positive matrix is positive.

Let the matrix of the matrix units be A such that

$$A = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Let $x = (-1 \;\; 2 \;\; -1 \;\; -1) \in \mathcal{H}$ be arbitrarily chosen and let $B(\mathcal{H}) \mapsto \mathbb{C}$, then

$$\begin{aligned} \langle Ax, x\rangle &= \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} -2 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle \\ &= 4. \end{aligned}$$

$$= \left\langle \begin{pmatrix} -2 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle$$
$$= 4.$$

Since $\langle Ax, x\rangle = 4 > 0,\ A$ is positive. In addition,

$$\|A\| = \sup |\langle Ax, x\rangle| = 4 < \infty.$$

Thus A is bounded.

Now, consider $(\phi_2)_2 : M_2(M_2(\mathcal{A})) \longrightarrow M_2(M_2(\mathcal{A}))$, such that

$$B = \phi_2\left(\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\right) = \begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $x = (-1\ \ 2\ -1\ -1) \in \mathcal{H}$ be arbitrarily chosen as before, then

$$\langle Bx, x\rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} -1 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle$$
$$= -2.$$

$\langle Bx, x\rangle = \langle \phi_2(M_2(\mathcal{A}))x, x\rangle < 0$. Thus B and hence ϕ_2 are not positive. However,

$$\|B\| = \sup |\langle Bx, x\rangle| = |-2| = 2 < \infty.$$

Thus, B is bounded and so $\|\phi_2\| = \sup \|\phi_2(M_2(\mathcal{A}))\| = 2$.

Thus, ϕ is positive but not 2-positive hence not completely positive. It turns out that ϕ_2 is bounded, hence completely bounded.

We check positivity and boundedness of C by induction.

Let

$$C = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $x = \begin{pmatrix} -1 & -1 & -1 & 2 & -1 & 2 & -1 & -1 \end{pmatrix} \in \mathcal{H}$ be arbitrarily chosen.

Then,

$$
\begin{aligned}
\langle Cx, x\rangle &= \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\rangle \\
&= 9 > 0.
\end{aligned}
$$

Thus, C is positive.

$\|C\| = \sup |\langle Cx, x\rangle| = 9 < \infty$.

Now, consider $(\phi_3)_3 : M_3(M_3(\mathcal{A})) \longrightarrow M_3(M_3(\mathcal{A}))$.

Then, let $D = \phi_3(M_3(\mathcal{A})) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

$$\langle Dx, x\rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\rangle = -3 < 0.$$

So, $\langle Dx, x\rangle = \langle \phi_3(M_3(\mathcal{A}))x, x\rangle = -3 < 0$, not positive. However, $\|D\| = \|\phi_3\| = 3$. So by induction $\langle \phi_n(M_n(\mathcal{A}))x, x\rangle = -n$. Hence not completely positive. In addition $\|\phi_n\| = n$. Thus, completely bounded for a finite n.

Thus a transpose map is completely bounded but not completely positive.

Chapter 4

CONCLUSIONS AND RECOMMENDATIONS

4.1 Conclusions

In chapter 1, the space $M_n(B(\mathcal{H}))$ of $n \times n$ matrices with entries from $B(\mathcal{H})$ is identified with the space $B\left(\mathcal{H}^{(n)}\right)$ of bounded linear operators on the n-dimensional Hilbert space $\mathcal{H}^{(n)}$. This identification gives us a unique norm that makes the $*$-algebra $M_n(B(\mathcal{H}))$ a C^*-algebra. Given an arbitrary C^*-algebra $\mathcal{A}$, by Gelfand Naimark Segul representation, $\mathcal{A}$ is a closed selfadjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This means that $M_n(\mathcal{A})$ is a closed selfadjoint subalgebra of the C^*-algebra $M_n(B(\mathcal{H})) \cong B\left(\mathcal{H}^{(n)}\right)$ and hence a C^*-algebra.

In chapter 2, its been shown that $[T_{ij}] \in M_n(B(\mathcal{H}))$ is positive if and only if $\langle [T_{ij}]x_j, x_i \rangle$ is positive. Some properties of the norm of completely positive maps have been investigated. Conditions for which a map is completely positive are discussed. A completely positive map satisfies the condition $\phi_n \geq 0$ for all $n \in \mathbb{N}$ and that $\phi = \phi^*$.

In chapter 3, conditions for complete boundedness are discussed. A map is completely bounded if its completely bounded norm is finite. In addition, a completely bounded and completely positive map satisfies the conditions $\|\phi(1)\| = \|\phi\| = \|\phi_n\| = \|\phi\|_{cb} = 1$ and $\|\phi\|_{cb} = \|\phi^*\|_{cb}$. This study has also shown that completely positive maps are all completely bounded for a finite-dimensional space. However, the converse does not always hold. Consequently, examples and Counter examples that not all positive maps are completely positive were illustrated. See Counter example [3.2.5]

Contributions of this thesis to mathematics are illustrated in : Theorem 2.1.7; Proposition 2.1.15; Proposition 2.3.2; Proposition 2.3.3; Proposition 3.0.5; Lemma 3.1.1; Proposition 3.1.2 and examples of positive (see 2.2), completely positive (see 2.4) and completely bounded maps (see 3.2) given.

4.2 Recommendations

From this study, although several properties of completely positive maps have been investigated, it is evident that completely positive maps is still an interesting and rich area of research in pure mathematics. Several other properties of completely positive maps could still be investigated. Areas such as the properties of non unital linear maps, the adjoining of a unit to them and determining whether they are completely positive or not, could also be of future interest to researchers.

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